Spring 2016 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. What percent of 45 is 36?
   Answer: **80** From $45(n/100) = 36$ we get $n = 100 \cdot 36/45 = 80$.

2. Cindy is 3 miles away from home. If she walks at a rate of 4 miles per hour, in how many minutes will she arrive at home?
   Answer: **45** Because $t = d/r = 3/4$ and $3/4$ of an hour is 45 minutes.

3. How many edges does a prism with hexagonal bases have?
   Answer: **18** Each base has six edges and then six edges connect the two bases together.

4. In a certain triangle the base is doubled and the height is tripled. What is the ratio of the area of the new triangle to the area of the orginal triangle?
   Answer: **6** The area is increased by a factor of $2 \cdot 3$ or 6.

5. Music streaming company Ossify charges a flat monthly fee of \$8 but charges 10 cents for each hour of listening over 50 hours for a given month. On the other hand, music streaming company Panaplex charges a flat monthly fee of \$6 but charges 14 cents for each hour of listening over 40 hours for a given month. How many total hours of listening would be required per month to make the total charges for that month from these companies turn out to be the same?
   Answer: **65** Solve $8 + (.1)(t - 50) = 6 + (.14)(t - 40)$ to obtain $t = 65$.

6. How many even postitive integers are factors of $3^5 - 1$?
   Answer: **3** Because $3^5 - 1 = 242 = 2 \cdot 11^2$ the only even factors are 2, $2 \cdot 11$, and $2 \cdot 11^2$.

7. Three times the complement of what angle is equal to the supplement of that angle?
   Answer: **45** Solve $3(90 - x) = 180 - x$ to get $x = 45$.

8. Hezy, Zeke, and Elias are running around a track in the same direction. Each of them runs at their own constant pace. Hezy is the fastest and passes Elias every 8 minutes. Meanwhile, Elias passes Zeke every 12 minutes. So how many seconds elapse between times Hezy passes Zeke?
   Answer: **288** Let Hezy run $n + 1$ laps in eight minutes while Elias runs n

laps. Since eight is two-thirds of 12 then in those eight minutes Zeke has run $n - 2/3$ laps. So in eight minutes Hezy has run $5/3$ of a lap more than Zeke. So in $8 \cdot 3/5$ minutes Hezy runs exactly one more lap than Zeke. Note $8 \cdot 3/5 \cdot 60 = 288$ seconds.

9. Admission to a zoo was \$ 20 per person when it was reduced to a new, lower rate. This caused the number of customers per day to increase by 40%. This in turn caused the amount collected by the zoo per day from admissions to increase by 12%. What is this new lower admission fee per person?
   Answer: **16** Let r be the new fee per person. Then $20(1)(1.12) = r(1.4)$ so $r = 22.4/1.4 = 16$.

10. In how many ways can the letters in DALLAS be arranged so that neither the A's nor the L's are next to each other?
    Answer: **84** Inclusion-Exclusion. Note $6!/(2!2!) - 2(5)4!/2! + 4! = 180 - 120 + 24 = 84$.

11. A group of 7th and 8th graders took the same math contest. The average score of all these students was 30. The average 7th grade score was 28 while the average 8th grade score was 33. What is the ratio of the number of 7th graders to the number of 8th graders?
    Answer: **3 : 2** From $28m + 33n = 30(m + n)$ or $3n = 2m$ or $m/n = 3/2$.

12. Find the 17th decimal place in the decimal expansion of the fraction 17/2200.
    Answer: **2** The repeating pattern settles in quickly so that after a bit the odd place decimals are all equal to 2.

13. Find the smallest value of the positive integer n so that the sum

$$1 + 2 + 3 + 4 + 5 + \cdots + n$$

    is divisible by 100.
    Answer: **24** We need $n(n+1)$ to be divisible by 200. Since n and $n+1$ differ by one their gcf is one, that is, they are relatively prime. That means that 25 either goes into n or into $n + 1$. In the first case the smallest n is 25. That does not turn out to work. Then if 25 goes into $n + 1$ the smallest n is 24 and that works.

14. Today my son is 1/5 of my age. Two years ago he was 1/7 of my age. In how many years from today will he be 1/3 of my age?
    Answer: **6** Solve $(1/5)n - (1/7)(n - 2) = 2$ to get $n = 30$ so I would be 30 now and my son 6. Then in six years, I would be 36 and my son 12.

15. Let $P = \{1, 4, 9, 16, 25, ...\}$ be the set of the squares of the positive integers. For how many elements p of P is $p + 144$ also an element of P?

Answer: **4** The correct values of p are $25, 81, 256$, and 1225. The idea is that consecutive squares differ by the odd integers, so we must write 144 as the sum of consecutive odd numbers. For example $144 = 11 + 13 + 15 + \cdots + 25$ (8 consecutive odd integers). But 11 separates the consecutive squares 25 and 36. Thus $25 + 144$ must be a perfect square as well.

Spring 2016 McNabb GDCTM Contest Answer Sheet

Name: Key School: ______ Date: ______

Contest Taking (Circle One):
Pre-algebra Algebra I Geometry Algebra II Pre-calculus Calculus

DIRECTIONS:

1. You are NOT permitted to use a calculator or any other electronic device. You have 60 minutes for Geometry, Algebra II, Pre-Calculus, or Calculus. You have 45 minutes for Pre-Algebra or Algebra One.
2. All answers should be in standard simplified from unless otherwise stated. All variables are assumed to stand for real numbers unless otherwise stated.

1. 80	6. 3	11. $\frac{3}{2}$ or 3:2
2. 45	7. 45	12. 2
3. 18	8. 288	13. 24
4. 6 or 6:1	9. 16	14. 6
5. 65	10. 84	15. 4

SCORE: ______

Spring 2016 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

1. Cindy is 3 miles away from home. If she walks at a rate of 4 miles per hour, in how many minutes will she arrive at home?
   Answer: **45** Because $t = d/r = 3/4$ and $3/4$ of an hour is 45 minutes.

2. Music streaming company Ossify charges a flat monthly fee of \$8 but charges 10 cents for each hour of listening over 50 hours for a given month. On the other hand, music streaming company Panaplex charges a flat monthly fee of \$6 but charges 14 cents for each hour of listening over 40 hours for a given month. How many total hours of listening would be required per month to make the total charges for that month from these companies turn out to be the same?
   Answer: **65** Solve $8 + (.1)(t - 50) = 6 + (.14)(t - 40)$ to obtain $t = 65$.

3. Find all solutions of
   $$20x^2 + 33x - 27 = 0$$
   Answer: $\mathbf{3/5}, \mathbf{-9/4}$ The quadratic factors as $(5x - 3)(4x + 9)$.

4. Three times the complement of what angle is equal to the supplement of that angle?
   Answer: **45** Solve $3(90 - x) = 180 - x$ to get $x = 45$.

5. Hezy, Zeke, and Elias are running around a track in the same direction. Each of them runs at their own constant pace. Hezy is the fastest and passes Elias every 8 minutes. Meanwhile, Elias passes Zeke every 12 minutes. So how many seconds elapse between times Hezy passes Zeke?
   Answer: **288** Let Hezy run $n + 1$ laps in eight minutes while Elias runs n laps. Since eight is two-thirds of 12 then in those eight minutes Zeke has run $n - 2/3$ laps. So in eight minutes Hezy has run $5/3$ of a lap more than Zeke. So in $8 \cdot 3/5$ minutes Hezy runs exactly one more lap than Zeke. Note $8 \cdot 3/5 \cdot 60 = 288$ seconds.

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Answer: **16** Let r be the new fee per person. Then $20(1)(1.12) = r(1.4)$ so $r = 22.4/1.4 = 16$.

7. A group of 7th and 8th graders took the same math contest. The average score of all these students was 30. The average 7th grade score was 28 while the average 8th grade score was 33. What is the ratio of the number of 7th graders to the number of 8th graders?
   Answer: $\mathbf{3 : 2}$ From $28m + 33n = 30(m + n)$ or $3n = 2m$ or $m/n = 3/2$.

8. Today my son is 1/5 of my age. Two years ago he was 1/7 of my age. In how many years from today will he be 1/3 of my age?
   Answer: **6** Solve $(1/5)n - (1/7)(n - 2) = 2$ to get $n = 30$ so I would be 30 now and my son 6. Then in six years, I would be 36 and my son 12.

9. Find one ordered triple of distinct positive integers (p, q, r) so that $p < q < r$ and
   $$\frac{3}{10} = \frac{1}{p} + \frac{1}{q} + \frac{1}{r}$$
   Write your answer as (p, q, r)
   Answer: $(\mathbf{4}, \mathbf{25}, \mathbf{100})$ We can start with $3/10 - 1/4 = 1/20$ (greedy algorigthm) which to too greedy. Then note that $1/20 = 5/100 = 4/100+1/100 = 1/25 + 1/100$. There are several other solutions as well.

10. If a, b, and c are positive integers satisfying $abc = 1560$, find the least possible value of $a + b + c$.
    Answer: **35** We want the factors as close to each other as possible. Those would be $10 \cdot 12 \cdot 13$.

11. Find the least value of x that satisfies
    $$|5x - 70| \leq |4x - 200|.$$
    Answer: $\mathbf{-130}$ The idea is try x as a negative number first. That makes the quantities inside the absolute values negative. So we would solve: $70 - 5x = 200 - 4x$.

12. Find the area of the parallelogram formed by the four lines
    $$y = 3x - 7$$
    $$y = 3x + 7$$
    $$y = 7x - 3$$
    $$y = 7x + 3$$

Answer: **21** Solve for the four vertices and use the shoestring formula. The four vertices are $(1,10)$, $(5/2,29/2)$, and the opposites of these.

13. I have only dimes and quarters in my pocket. There are b coins in all, and all together they are worth c cents. In terms of b and c, how many quarters do I have in my pocket?
Answer: $(\mathbf{c}-\mathbf{10b})/\mathbf{15}$ Solve the system $x+y=b$ and $10x+25y=c$ for y.

14. When the cubic polynomial x^3-x^2+kx-2 is divided by $x-3$ the remainder is k. Find the value of the constant k.
Answer: $\mathbf{-8}$ When $x=3$ is substituted into the polynomial the result is k. Thus solve $27-9+3k-2=k$

15. Let $P=\{1,4,9,16,25,....\}$ be the set of the squares of the positive integers. For how many elements p of P is $p+144$ also an element of P?
Answer: **4** The correct values of p are $25, 81, 256$, and 1225. The idea is that consecutive squares differ by the odd integers, so we must write 144 as the sum of consecutive odd numbers. For example $144 = 11+13+15+\cdots+25$ (8 consecutive odd integers). But 11 separates the consecutive squares 25 and 36. Thus $25+144$ must be a perfect square as well.

Spring 2016 McNabb GDCTM Contest Answer Sheet

Name: Key School: ______ Date: ______

Contest Taking (Circle One):
Pre-algebra Algebra I Geometry Algebra II Pre-calculus Calculus

DIRECTIONS:

1. You are NOT permitted to use a calculator or any other electronic device. You have 60 minutes for Geometry, Algebra II, Pre-Calculus, or Calculus. You have 45 minutes for Pre-Algebra or Algebra One.
2. All answers should be in standard simplified from unless otherwise stated. All variables are assumed to stand for real numbers unless otherwise stated.

1. 45	6. 16	11. -130
2. 65	7. $\frac{3}{2}$ or 3:2	12. 21
3. $3/5$ and $-9/4$ (need both)	8. 6	13. $\frac{c-10b}{15}$
4. 45	9. One of: (4, 25, 100), (6, 10, 30) (4, 21, 420), (5, 11, 110) others possible	14. -8
5. 288	10. 35	15. 4

SCORE: ______

Spring 2016 McNabb GDCTM Contest
Geometry Solutions

NO Calculators Allowed

1. How many edges does a prism with hexagonal bases have?
Answer: **18** Each base has six edges and then six edges connect the two bases together.

2. Find the ratio of the square of the circumference of a circle to the area of that same circle.
Answer: $\mathbf{4\pi}$ Simplify $(2\pi r)^2/(\pi r^2)$.

3. Find the value of k for which the point $(3k-1, k)$ lies on the line $7x - 3y = 2$.
Answer: **1/2** Sub in to get $7(3k-1) - 3k = 2$ and solve for k.

4. Three times the complement of what angle is equal to the supplement of that angle?
Answer: **45** Solve $3(90 - x) = 180 - x$ to get $x = 45$.

5. In a certain triangle the base is doubled and the height is tripled. What is the ratio of the area of the new triangle to the area of the orginal triangle?
Answer: **6** The area is increased by a factor of $2 \cdot 3$ or 6.

6. The front face of a rectangular box has area 72. Its left face has area 48 while its top face has area 96. Find the volume of the box.
Answer: **576** One could try to seek integers for the side lengths to make these areas of the faces happen. Or else observe that $V = \sqrt{72 \cdot 48 \cdot 96}$ since each dimension appears twice in that product.

7. Find the area of a triangle with sides of length 9, 10, and 11.
Answer: $\mathbf{30\sqrt{2}}$ Use Heron's formula.

8. Find the area of the parallelogram formed by the four lines

$$\begin{aligned} y &= 3x - 7 \\ y &= 3x + 7 \\ y &= 7x - 3 \\ y &= 7x + 3 \end{aligned}$$

Answer: **21** Solve for the four vertices and use the shoestring formula. The four vertices are $(1, 10)$, $(5/2, 29/2)$, and the opposites of these.

9. Find the coordinates of the center of the circumcircle of the triangle whose vertices are given by: $(2, 0), (0, 2)$, and $(10, 0)$.
   Answer: $(\mathbf{6}, \mathbf{6})$ The perpendicular bisectors of two of the sides are $x = 6$ and $y = x$.

10. In $\triangle ABC$, the bisector of $\angle A$ meets side BC at point D. Find the ratio of the area of $\triangle ABD$ to the area of $\triangle ADC$ if $AB = 13$ and $AC = 17$.
    Answer: $\mathbf{13} : \mathbf{17}$ Apply the Angle Bisector Theorem and recall that the ratios of the areas of triangles with the same height are as their bases.

11. Let points A, B, C, and D lie evenly spaced on a line in that order. On BC as base an equilateral triangle BCP is drawn. If $AB = 12$, determine AP.
    Answer: $\mathbf{12\sqrt{3}}$ The altitude from P is $6\sqrt{3}$. Then by the Pythagorean Theorem $(AP)^2 = (6\sqrt{3})^2 + 18^2$.

12. Two congruent circles have a common external tangent of length 20 and a common internal tangent of length 18. What is the common radius of the two circles?
    Answer: $\sqrt{\mathbf{19}}$ The two centers are 20 apart. The triangle with vertices as the center of one circle, the midpoint between the two centers, and the point of intersection of the internal tangent with the circle we took the center of is a right triangle with hypotenuse 10, one leg the radius of that circle, and the other leg 9.

13. Two circles intersect at points A and B. The tangents to the two circles at point A meet at right angles. The radius of the smaller circle is 8 and the radius of the larger circle is 15. Find the length of the AB.
    Answer: $\mathbf{240/17}$ The triangle with vertices the two centers and A is a right triangle with sides 8, 15, and 17. The altitude to the hypotenuse is $120/7$ and AB is twice this.

14. A convex pentagon has side lengths in cyclic order as: $17, 6, 13, 26$, and 4. The sides of lengths 26 and 6 are parallel, and the sides of lengths 26 and 4 are perpendicular. What is the area of the pentagon?
    Answer: $\mathbf{222}$ This pentagon decomposes into: a 15 by 4 rectangle, an $8 - 15 - 17$ right triangle, a 6 by 12 rectangle, and a $5 - 12 - 13$ right triangle. These areas add up to 222.

15. In $\triangle ABC$, $AB = AC$, and P and Q are the midpoints respectively of AB and AC. Extend BC to point D so that $CD = BC$. Let PD meet AC at point R. Find the ratio of QR to AC.

Answer: **1 : 6** Note $\triangle PQR$ is similar to $\triangle DCR$ with ratio of similitude 2 by the Midline Theorem. Thus QR is half of RC, making QR one-third of QC.

Spring 2016 McNabb GDCTM Contest Answer Sheet

Name: Key School: ______ Date: ______

Contest Taking (Circle One):
Pre-algebra Algebra I Geometry Algebra II Pre-calculus Calculus

DIRECTIONS:

1. You are NOT permitted to use a calculator or any other electronic device. You have 60 minutes for Geometry, Algebra II, Pre-Calculus, or Calculus. You have 45 minutes for Pre-Algebra or Algebra One.
2. All answers should be in standard simplified from unless otherwise stated. All variables are assumed to stand for real numbers unless otherwise stated.

1. 18	6. 576	11. $12\sqrt{3}$
2. 4π or $4\pi:1$	7. $30\sqrt{2}$	12. $\sqrt{19}$
3. $1/2$	8. 21	13. $240/17$
4. 45	9. $(6,6)$	14. 222
5. 6 or $6:1$	10. $\frac{13}{17}$ or $13:17$	15. $1:6$ or $\frac{1}{6}$

SCORE: ______

Spring 2016 McNabb GDCTM Contest
Algebra II Solutions

NO Calculators Allowed

1. A thirteen foot tree is growing at a rate of three feet per year while a forty-one foot tree is growing at a rate of two feet per year. In how many years will the two trees be the same height?
   Answer: **28** The shorter tree gains one foot per year so the difference ends up taking $41 - 13 = 28$ years to close.

2. The front face of a rectangular box has area 72. Its left face has area 48 while its top face has area 96. Find the volume of the box.
   Answer: **576** One could try to seek integers for the side lengths to make these areas of the faces happen. Or else observe that $V = \sqrt{72 \cdot 48 \cdot 96}$ since each dimension appears twice in that product.

3. Solve for x:
   $$\log_2 8 - \log_3 9 = \log_5 x$$
   Answer: **5** Same as $3 - 2 = 1 = \log_5 x$ so $x = 5$.

4. Find all solutions of
   $$20x^2 + 33x - 27 = 0$$
   Answer: $\mathbf{3/5, -9/4}$ The quadratic factors as $(5x - 3)(4x + 9)$.

5. Find the sum
   $$1 + i + i^2 + i^3 + i^4 + i^5 + \cdots + i^{2016}$$
   where $i = \sqrt{-1}$.
   Answer: **1** Every four consecutive terms add up to 0. Then $i^{2019} = 1$ is left over. Or else use the formula for the sum of a finite geometric series.

6. When the cubic polynomial $x^3 - x^2 + kx - 2$ is divided by $x - 3$ the remainder is k. Find the value of the constant k.
   Answer: $\mathbf{-8}$ When $x = 3$ is substituted into the polynomial the result is k. Thus solve $27 - 9 + 3k - 2 = k$

7. Find the maximum value of $x + y$ given that
   $$3x + 11y \leq 198$$
   $$5x + y \leq 70$$

Answer: **26** The two boundary lines meet at $(11, 15)$ where the maximum occurs.

8. Find the maximum number of regions of the plane formed by three ellipses lying in that plane.
Answer: **14** Carefully draw overlapping horizontal, vertical, and $45°$ tilted ellipses, similar to logo for the Atomic Energy Comission. There are then 14 regions including the unbounded region.

9. Find the minimum value of the function $f(x, y)$ where

$$f(x, y) = |20 - x| + |x - y| + |y - 50|.$$

Answer: **30** To minimize the sum choose the points in the order $20 < x < y < 50$.

10. Let S_n equal the sum of the first n terms of an arithmetic sequence. If $S_{20} = 180$ and $S_{40} = 500$, find the value of S_{60}.
Answer: **960** In general for arithmetic series, $S_{3n} = 3(S_{2n} - S_n)$.

11. A *lattice point* in the plane is a point such that both of its coordinates are integers. How many such lattice points lie on the curve $x^2 + 2y^2 = 81$?
Answer: **10** The ten solutions are $(\pm 3, \pm 6)$, $(\pm 7, \pm 4)$, $(9, 0)$, and $(-9, 0)$.

12. Find the sum of all the solutions of the equation

$$\frac{1}{x} - \frac{1}{x+1} = \frac{1}{3x+15}$$

Answer: **2** The solutions are 5 and -3 using usual textbook method.

13. Find the sum of the cubes of the roots of

$$x^3 - 11x^2 + 9 = 0$$

Answer: **1304** Let the roots be r, s, and t. Then $r + s + t = 11$ so $r^2 + s^2 + t^2 = (r + s + t)^2 = 121$ because $rs + st + tr = 0$. Then $r^3 + s^3 + t^3 = 11(r^2 + s^2 + t^2) - 27 = 11 \cdot 121 - 27 = 1304$.

14. In how many ways can 2016 be written as the sum of two or more consecutive integers?
Answer: **11** For each odd factor of 2016 greater than one there are that many

consecutive integers that add up to 2016. This gives five ways. Then for each odd factor (including one) there is an even number of consecutive integers that add up to 2016. This gives six more ways.

15. For some constants a, b, and c, we have that

$$p(x) = x^3 - ax^2 + bx - c$$
$$p(x) = (x-a)(x-b)(x-c)$$

Find the value of $p(4)$.
Answer: **64** or **75** Intended solution is 75 but if $a = b = c = 0$ then 64 works too. (Should have said "non-zero" constants.) Assuming non-zero, then from $-abc = -c$ we get $ab = 1$. From $a+b+c = a$ we get $b+c = 0$. From $ab+bc+ac = bc+a(b+c) = bc = b$ we get $c = 1$. Then $b = -1$ and $a = -1$. So $p(x) = x^2 + x^2 - x - 1$ and $p(4) = 64 + 16 - 4 - 1 = 75$.

SPRING 2016 McNABB GDCTM CONTEST ANSWER SHEET

Name: Key School: ______ Date: ______

Contest Taking (Circle One):
Pre-algebra Algebra I Geometry Algebra II Pre-calculus Calculus

DIRECTIONS:

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2. All answers should be in standard simplified from unless otherwise stated. All variables are assumed to stand for real numbers unless otherwise stated.

1. 28	6. −8	11. 10
2. 576	7. 26	12. 2
3. 5	8. 14	13. 1304
4. $-\frac{9}{4}$ and $\frac{3}{5}$ (need both)	9. 30	14. 11
5. 1	10. 960	15. 64 or 75 (need only one)

SCORE: ______

Spring 2016 McNabb GDCTM Contest
PreCalculus Solutions

NO Calculators Allowed

1. Find the prime factorization of $3^8 - 1$.
   Answer: $\mathbf{2^5 \cdot 5 \cdot 41}$ Use difference of squares $3^8 - 1 = (3^4+1)(3^4-1) = 82 \cdot 80$.

2. Find a pair of positive integers (m, n) that satisfy $17m - 19n = 1$.
   Answer: $(\mathbf{9}, \mathbf{8})$ You can unravel the Euclidean Algorithm that finds the gcf of 17 and 19. Or do trial and error.

3. Find the maximum value of $11\cos\theta - 2\cos^2\theta$.
   Answer: **9** Same as maximum of $y = 11x - 2x^2$ over the interval $[-1, 1]$. This interval does not include the vertex so the maximum occurs at $x = 1$ and is equal to $11 - 2 = 9$.

4. Ten chairs are set up in a row. In how many ways can three people occupy the chairs so that no two sit next to each other?
   Answer: **336** Inclusion-Exclusion: $3!(\binom{10}{3} - 9 \cdot 8 + 8) = 6(120 - 72 + 8) = 336$

5. In how many ways can a class of 12 students be split into three groups of four students each?
   Answer: **5775** Compute as
   $$\frac{\binom{12}{4} \cdot \binom{8}{4}}{3!}$$
   We divide because the groups have no order to them.

6. For all $x \neq 0$, let
   $$2f(x) + 5xf(1/x) = 3x + 2$$
   Find x if $f(x) = 7$.
   Answer: **34** Replace x by $1/x$ and solve the resulting system to get $f(x) = (4x + 11)/21$.

7. The longer base of an isosceles trapezoid is equal to a diagonal of the trapezoid. The shorter base of the trapezoid is equal to the altitude of the trapezoid. Find the ratio of the shorter base to the longer base.
   Answer: **3 : 5** Write the longer base as $3 + 2x$. Then apply the Pythagorean Theorem to the right triangle with legs 3, $3 + x$, and $3 + 2x$.

8. Find the number of ways to make change for 2 dollars using nickels, dimes, and quarters.
Answer: **97** Make an organized table. Or there exists a quadratic function of the number of multiples of dollars that change is made for.

9. Passwords for a certain device must use only the capital letters A, B, or C. The passwords must be exactly of length 8 and each of those three capital letters must be used at least once. How many such passwords are there?
Answer: **5796** We use includsion-exclusion:

$$3^8 - 3(2^8) + 3(1^8) = 5796$$

10. Let
$$z + \frac{1}{z} = 2\cos(15^\circ)$$
Find an integer n such that $0 < n < 90$ and
$$z^2 + \frac{1}{z^2} = 2\cos(n^\circ)$$
Answer: **30** Note that $(z + 1/z)^2 = z^2 + 2 + 1/z^2$. Thus
$$z^2 + 1/z^2 = 4\cos^2(15^\circ) - 2 = 2(2\cos^2 15^\circ - 1) = 2\cos 30^\circ.$$
So $n = 30$.

11. Find a 2×2 matrix M with integer entries that satisfies the equation:
$$M^2 = \begin{pmatrix} 5 & -4 \\ 4 & -3 \end{pmatrix}$$
Answer: Either this matrix or its opposite:
$$M = \begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$$

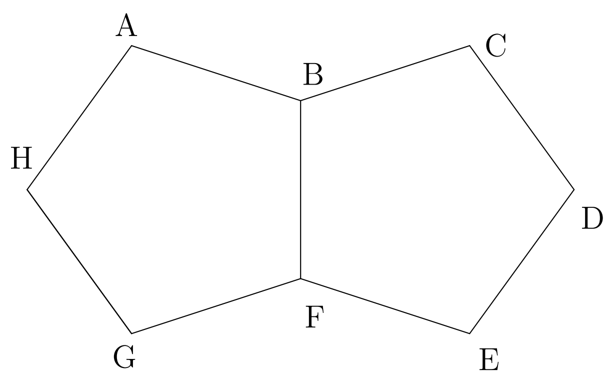
12. Let the function $f(x, y)$ satisfy the recursive rules
$$f(x, y+1) = f(f(x, y), y) + 4$$
$$f(x, 0) = x$$
Calculate the value of $f(5, 5)$.
Answer: **129** After working some small cases conjecture that
$$f(x, y) = x + (2^y - 1) \cdot 4$$
which may be shown by induction.

13. Evaluate

$$\frac{\cos 87^\circ}{\sin 1^\circ} - \frac{\sin 87^\circ}{\cos 1^\circ}$$

Answer: **2** Form common denominators and subtract. Then use trig identities to see the fraction is equivalent to $\cos 88^\circ/((1/2)\sin 2^\circ$. But $\sin 2^\circ = \cos 88^\circ$.

14. Two regular pentagons, both of side length 2, are glued togther at one edge to form a non-convex octogon $ABCDEFGH$ as shown. What is the value of $(EG)^2$? Your answer must be in the form $a + b\sqrt{c}$ where a, b, and c are positive integers and c has no perfect square factors greater than one. Answer:



$\mathbf{10 + 2\sqrt{5}}$ Use the law of cosines on $\triangle EFG$.

15. A *lattice point* in the plane is a point such that both of its coordinates are integers. How many such lattice points lie on the curve $x^2 + 2y^2 = 81$?
Answer: **10** The ten solutions are $(\pm 3, \pm 6)$, $(\pm 7, \pm 4)$, $(9, 0)$, and $(-9, 0)$.

Spring 2016 McNabb GDCTM Contest Answer Sheet

Name: Key School: ______ Date: ______

Contest Taking (Circle One):
Pre-algebra Algebra I Geometry Algebra II Pre-calculus Calculus

DIRECTIONS:

1. You are NOT permitted to use a calculator or any other electronic device. You have 60 minutes for Geometry, Algebra II, Pre-Calculus, or Calculus. You have 45 minutes for Pre-Algebra or Algebra One.
2. All answers should be in standard simplified from unless otherwise stated. All variables are assumed to stand for real numbers unless otherwise stated.

1. $2^5 \cdot 5 \cdot 41$

2. $(9, 8)$

3. 9

4. 336

5. 5775

6. 34

7. $\frac{3}{5}$

8. 97

9. 5796

10. 30

11. $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ or its opposite

12. 129

13. 2

14. $10+2\sqrt{5}$

15. 10

SCORE: ______

Spring 2016 McNabb GDCTM Contest
Calculus Solutions

NO Calculators Allowed

1. In how many ways can the letters in DALLAS be arranged so that neither the A's nor the L's are next to each other?
   Answer: **84** Inclusion-Exclusion. Note $6!/(2!2!) - 2(5)4!/2! + 4! = 180 - 120 + 24 = 84$.

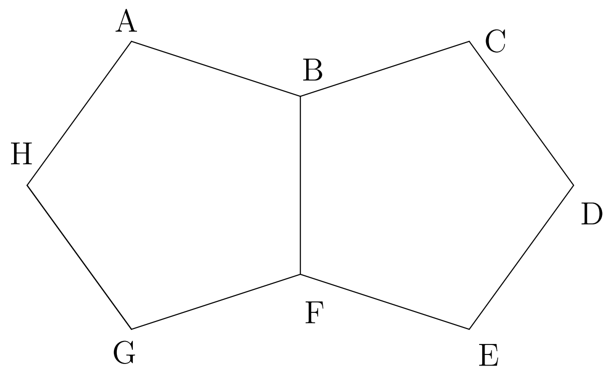
2. Hezy, Zeke, and Elias are running around a track in the same direction. Each of them runs at their own constant pace. Hezy is the fastest and passes Elias every 8 minutes. Meanwhile, Elias passes Zeke every 12 minutes. So how many seconds elapse between times Hezy passes Zeke?
   Answer: **288** Let Hezy run $n+1$ laps in eight minutes while Elias runs n laps. Since eight is two-thirds of 12 then in those eight minutes Zeke has run $n - 2/3$ laps. So in eight minutes Hezy has run $5/3$ of a lap more than Zeke. So in $8 \cdot 3/5$ minutes Hezy runs exactly one more lap than Zeke. Note $8 \cdot 3/5 \cdot 60 = 288$ seconds.

3. If a, b, and c are positive integers satisfying $abc = 1560$, find the least possible value of $a + b + c$.
   Answer: **35** We want the factors as close to each other as possible. Those would be $10 \cdot 12 \cdot 13$.

4. Find the sum
   $$1 + i + i^2 + i^3 + i^4 + i^5 + \cdots + i^{2016}$$
   where $i = \sqrt{-1}$.
   Answer: **1** Every four consecutive terms add up to 0. Then $i^{2019} = 1$ is left over. Or else use the formula for the sum of a finite geometric series.

5. Find the maximum number of regions of the plane formed by three ellipses lying in that plane.
   Answer: **14** Carefully draw overlapping horizontal, vertical, and 45° tilted ellipses, similar to logo for the Atomic Energy Comission. There are then 14 regions including the unbounded region.

6. Two regular pentagons, both of side length 2, are glued togther at one edge to form a non-convex octogon $ABCDEFGH$ as shown. What is the value of $(EG)^2$? Your answer must be in the form $a + b\sqrt{c}$ where a, b, and c are



positive integers and c has no perfect square factors greater than one. Answer: $\mathbf{10 + 2\sqrt{5}}$ Use the law of cosines on $\triangle EFG$.

7. Find the maximum value of $11\cos\theta - 2\cos^2\theta$.
   Answer: **9** Same as maximum of $y = 11x - 2x^2$ over the interval $[-1, 1]$. This interval does not include the vertex so the maximum occurs at $x = 1$ and is equal to $11 - 2 = 9$.

8. For what value of n is it true that

$$\int_0^n x^2\,dx = 9$$

   ?
   Answer: **3** Solve $9 = n^3/3$.

9. Find the coordinates of a point on the curve $x^2 + xy + y^2 = 3$ at which the curve has a horizontal tangent line.
   Answer: $(\mathbf{1}, -\mathbf{2}), (-\mathbf{1}, \mathbf{2})$ Use Implicit Differentiation, set numerator equal to zero, then sub back into the equation of the curve.

10. Evaluate

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n \frac{x^4}{3x^4 + 1}\,dx$$

   Answer: $\mathbf{1/3}$ Apply L'Hospital's Rule. Or else note that the horizontal asymptote is $y = 1/3$ so the region more and more resembles a rectangle with height $1/3$.

11. Find the total area enclosed by the polar graph $r^2 = 18\cos(2\theta)$.
   Answer: **18** The area equals $4\int_0^{\pi/4} (1/2) r^2\,d\theta$.

12. Evaluate

$$\int_1^{64} \frac{1}{\sqrt{x}(\sqrt{x}+\sqrt[3]{x})}\,dx$$

Answer: $\mathbf{6}\ln(\mathbf{3/2})$ Use the u-substitution $u = x^{1/6}$.

13. Let

$$f(x) = \frac{2}{x^2+10x+24}$$

Find the value of the sixth derivative of $f(x)$ at the point $x=-5$.
Answer: $-\mathbf{1440}$ Apply partial fractions before taking the derivatives.

14. Evaluate

$$\sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2^n}$$

Answer: **16** Use the Maclaurin series for $f(x) = 2/(1-x)^3$ and let $x = 1/2$.

15. Evaluate

$$\int_0^{\infty} \frac{\tan^{-1}(ex)-\tan^{-1}(x)}{x}\,dx$$

Answer: $\pi/\mathbf{2}$ Use Feynman's trick. That is we let

$$f(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)-\tan^{-1}(x)}{x}\,dx$$

Then $f'(a) = \int_0^{\infty} \dfrac{x}{x(1+a^2x^2)}\,dx = \int_0^{\infty} \dfrac{1}{1+a^2x^2}\,dx = (1/a)(\pi/2)$. So $f(a) = (\ln a)(\pi/2) + C$. But $f(1) = 0$, so $C = 0$ and $f(a) = (\ln a)(\pi/2)$. Thus $f(e) = \pi/2$.

SPRING 2016 MCNABB GDCTM CONTEST ANSWER SHEET

Name: Key School: __________ Date: __________

Contest Taking (Circle One):
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1. 84	6. $10+2\sqrt{5}$	11. 18
2. 288	7. 9	12. $6\ln(3/2)$ or equivalent
3. 35	8. 3	13. -1440
4. 1	9. $(1,-2)$ or $(-1,2)$ (need only one)	14. 16
5. 14	10. $\frac{1}{3}$	15. $\frac{\pi}{2}$

SCORE:__________