Spring 2013 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. If 10% of a is b what is 10% of b?

   (A) $100a$ **(B)** $10a$ **(C)** a **(D)** $.1a$ **(E)** $.01a$

   Solution: (E) From $.1a = b$ we have $.1b = .01a$.

2. If 10 carpenters can build 10 cabinets in 10 days how many days does it take 20 carpenters to build 20 cabinets?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

   Solution: B Twice as many carpenters will build twice as many cabinets in the same amount of time.

3. Express the fraction

$$\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5}}}$$

   in lowest terms.

   (A) $1/8$ **(B)** $11/15$ **(C)** $15/21$ **(D)** $16/21$ **(E)** $21/16$

   Solution: D First, $3 + 1/5 = 16/5$. Then $1 + 5/16 = 21/16$. So the answer is $16/21$.

4. The square root of 20000 lies between

   (A) 130 and 131 **(B)** 140 and 141 **(C)** 141 and 142 **(D)** 142 and 143 **(E)** 10,000 and 10,001

   Solution: C We have $\sqrt{20000} = 100\sqrt{2} \approx 141.4$.

5. The last 6 digits of 13^{426} are 000009. What is the sum of the last 6 digits of 13^{1704}?

   (A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

   Solution: A Since $(426) * 4 = 1704$ we have to examine $9^4 = 81^2 = 6561$. Because there are 5 zeros in front of 9 then 13^{1704} will easily end in 006561. The sum of these digits is 18.

6. In the repeating decimal $0.\overline{71771}$, in which decimal place does the 2013th 7 appear?

   (A) 671st **(B)** 2014th **(C)** 2015th **(D)** 3354th **(E)** 3355th

   Solution: D 2013 divided by 3 gives 671, so the last 7 in the 671st group of 71771 is the 2013th 7. It occurs in the $670 \cdot 5 + 4$th spot, which is the 3354th spot.

7. How many positive factors does 2013 have?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

   Solution: B From $2013 = 3 \cdot 11 \cdot 61$, there are $2 \cdot 2 \cdot 2 = 8$ factors of 2013.

8. If the integer $\underline{4400b074}$ is divisible by 101, what must the digit b equal?

   (A) 0 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

   Solution: C A rule for divisibility by 101 is to add and subtract groups of two digits, so that this number is divisible by 101 if and only if the original is. So $44 - 00 + b0 - 74 = b0 - 30$. Thus $b = 3$.

9. The points A, B, C, and D are the vertices of a unit square. How many squares in the plane of these points (including $ABCD$ itself) have two or more of them as vertices?

   (A) 4 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 13

   Solution: (E) Not counting the original square, each pair of consecutive vertices belongs to two distinct square, one where these two are consecutive and the other where these two are diagonally opposite. Each pair of diagonally opposite vertices belong to two squares other than the original, where they are consecutive in those. Thus there are $4 \cdot 2 + 2 \cdot 2 + 1 = 13$ such squares. (Once at least three of the original 4 points are included they can only generate the original square.)

10. Using estimation, the number of digits in the number 2^{50} is

(A) between 6 and 10 inclusive
(B) between 11 and 15 inclusive
(C) between 16 and 20 inclusive
(D) between 21 and 25 inclusive
(E) 26 or more

Solution: C Since $2^{10} = 1024$ then $2^{50} > 10^{15}$ and so 2^{50} has at least 16 digits. At the same time $2^{50} < (2000)^5 = 32 * 10^{15}$ and this last number has 17 digits.

11. Four roses and two tulips are to be arranged in a circle. Two such arrangements are considered to be the same if and only if each can be rotated into the other. How many distinct arrangements are possible?

(A) 3 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 15

Solution: A The only possible cases are: (1) the two tulips occur together; (2) the two tulips are separated by a single rose on one side and 3 on the other; (3) the two tulips are separated by two roses on either side.

12. How many seconds are there in exactly six weeks?

(A) 7! **(B)** 8! **(C)** 9! **(D)** 10! **(E)** 12!

Solution: (D) Note that $6 \cdot 7 \cdot 24 \cdot 60 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 180 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 120 = 10!$

13. The product of a certain integer and 180 is a perfect square. That certain integer must be divisible by

(A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

Solution: D Say $180n = m^2$. Then divide both sides by 36 so that $5n = k^2$ where k is a factor of m. So k is divisible by 5 and hence n must be divisible by 5.

14. A rectangle with area 125 has its sides in the ratio of 4 : 5. What is the perimeter of this rectangle?

(A) 18 **(B)** 22.5 **(C)** 36 **(D)** 45 **(E)** 54

Solution: (D) Let the sides be $4a$ and $5a$. So the perimeter equals $18a$. But $20a^2 = 125$ or $a = 5/2$. Thus $18a = 9 \cdot 5 = 45$.

15. I have two numbers in mind. The first number leaves a remainder of 4159 when divided by 5153 while the second number leaves a remainder of 5149 when divided by 5153. What is the remainder when the sum of these numbers is divided by 5153?

(A) 3135 **(B)** 3455 **(C)** 4144 **(D)** 4155 **(E)** 4344

Solution: D The sum of the two remainders is 9308 which must be reduced by 5153 so that the true remainder of the sum will be 4155.

16. In the following arrangement of the positive integers, in which column, counting from left to right, does 7021 appear?

$$
\begin{array}{cccc}
 & & & \cdot \\
 & & 5 & \cdots \\
 & 2 & 6 & \cdots \\
1 & 3 & 7 & \cdots \\
 & 4 & 8 & \cdots \\
 & & 9 & \cdots \\
 & & & \cdot
\end{array}
$$

(A) 43 **(B)** 51 **(C)** 52 **(D)** 84 **(E)** 99

Solution: (D) Note that the last entry of each column forms the perfects squares in order. Since $83^2 < 7021 < 84^2$, 7021 occurs in the 84th column.

17. Each face of a cube is numbered with a positive integer in such a way that the numbers on pairs of faces sharing an edge differ by at least two. What is the minimum possible sum of six such integers?

(A) 12 **(B)** 15 **(C)** 18 **(D)** 24 **(E)** 27

Solution: (C) The minimum sum occurs when opposite pairs of faces share the numbers 1, 3, and 5, giving a sum of $2(1+3+5) = 18$.

18. The value of

$$1 + 2 + 3 + 4 - 5 + 6 + 7 + 8 + 9 - 10 + \cdots + 46 + 47 + 48 + 49 - 50$$

is equal to

(A) 600 **(B)** 650 **(C)** 725 **(D)** 750 **(E)** 800

Solution: (C) The sum is $5 + 20 + 35 + \cdots + 140 = (145)5 = 725$.

Spring 2013 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. If 10 carpenters can build 10 cabinets in 10 days how many days does it take 20 carpenters to build 20 cabinets?

   (A) 5 **(B)** 10 **(C)** 15 **(D)** 20 **(E)** 25

   Solution: B Twice as many carpenters will build twice as many cabinets in the same amount of time.

2. The last 6 digits of 13^{426} are 000009. What is the sum of the last 6 digits of 13^{1704}?

   (A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

   Solution: A Since $(426) * 4 = 1704$ we have to examine $9^4 = 81^2 = 6561$. Because there are 5 zeros in front of 9 then 13^{1704} will easily end in 006561. The sum of these digits is 18.

3. How many seconds are there in exactly six weeks?

   (A) 7! **(B)** 8! **(C)** 9! **(D)** 10! **(E)** 12!

   Solution: (D) Note that $6 \cdot 7 \cdot 24 \cdot 60 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 180 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 120 = 10!$

4. If the integer $\underline{4400b074}$ is divisible by 101, what must the digit b equal?

   (A) 0 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 8

   Solution: C A rule for divisibility by 101 is to add and subtract groups of two digits, so that this number is divisible by 101 if and only if the original is. So $44 - 00 + b0 - 74 = b0 - 30$. Thus $b = 3$.

5. A rectangle with area 125 has its sides in the ratio of 4 : 5. What is the perimeter of this rectangle?

   (A) 18 **(B)** 22.5 **(C)** 36 **(D)** 45 **(E)** 54

   Solution: (D) Let the sides be $4a$ and $5a$. So the perimeter equals $18a$. But $20a^2 = 125$ or $a = 5/2$. Thus $18a = 9 \cdot 5 = 45$.

6. For how many positive integers n does $n!$ end in exactly eleven zeros?

   (A) 0 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 11

   Solution: A When tracking these zeros note that 45! ends in 10 zeros exactly, while 50! ends in 12 zeros exactly. Thus no factorial ends in exactly 11 zeros.

7. Which of the following equations has exactly two solutions over the real numbers?

   (A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
   (D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

   Solution: (D) Its solution set is $\{-12, 12\}$. The others do not have exactly two real solutions.

8. When a cyclist gets a puncture she has just completed three-fourths of her route. She finishes her route by walking. If she spent twice as much time walking as biking, how many times faster does she bike than walk?

   (A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

   Solution: (E) Let k be how many times faster she bikes than walks. Let w be how far she walked. Then $1/w = 2 \cdot 3/(kw)$ which reduces to $k = 6$.

9. Given the three points $(2013, -1863)$, $(1776, -1812)$, and $(1181, -1492)$ in the coordinate plane, a fourth point (a, b) is called a *complementing* point if it along with the given three points form the vertices of a parallelogram. Find the sum of all the coordinates of all the complementing points of the given three points.

   (A) -197 **(B)** 0 **(C)** 216 **(D)** 631 **(E)** 783

   Solution: (A) For any triangle, let its vertices V_1, V_2, and V_3, be viewed as vectors each with initial point at the origin. Then there are 3 complementing points $P_1 = V_2 + V_3 - V_1$, $P_2 = V_1 + V_3 - V_2$, and $P_3 = V_1 + V_2 - V_3$ giving the nice result that $P_1 + P_2 + P_3 = V_1 + V_2 + V_3$. Thus our answer is $2013 + 1776 + 1181 - 1863 - 1812 - 1492 = -197$.

10. The sum of a set of numbers is the sum of all the numbers in that set. How many subsets of the set $\{1,2,3,4,5,6,7\}$ have a sum of 12?

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

Solution: (E) There are exactly 8 such subsets: $\{7,5\},\{7,4,1\},\{7,3,2\},\{6,5,1\},\{6,4,2\},\{6,1,2,3\},\{5,4,3\},\{5,4,2,1\}$.

11. An amount of \$10000 dollars is deposited in an account for one year at an interest rate of x percent per year compounded twice a year. If, at the end of the year, \$10404 is in the account, then x is

(A) 3.9 **(B)** 4 **(C)** 4.1 **(D)** 7.8 **(E)** 8

Solution: (B) Let $y = x/100$. We have $10000(1+y/2)^2 = 10404$ so that $1+y/2 = \sqrt{1.0404} = 1.02$, so that $y/2 = .02$ or $y = .04$. Thus $x = 4$.

12. Which transformation never changes the median of a list of a dozen distinct positive integers?

(A) adding 6 to each number in the list
(B) adding 3 to each of the three smallest numbers in the list
(C) subtracting 4 from each of the four largest numbers in the list
(D) doubling each number in the list
(E) taking the reciprocal of each number in the list

Solution: (B) Adding 3 to each of the three smallest numbers in the list can change at most the five smallest numbers or add duplicates up to the 6th smallest number in the list, since the original list has all distinct integers. Thus the sixth and seventh cannot themseleves change under this transformation, which of course preserves the median. All the other transformations can or must change the median.

13. One factor of $14x^2 + 37x + 24$ is

(A) $2x+1$ **(B)** $14x+3$ **(C)** $2x+7$ **(D)** $7x+8$ **(E)** $7x+3$

Solution: D The quadratic factors as $(7x+8)(2x+3)$.

14. A purse may only contain pennies, nickels, dimes, and quarters but does not have to contain any particular type of coin, except as demanded in meeting the following conditions: the average value of the coins in the purse is 16 cents; if one more quarter were added to it the average value would rise to 17 cents. How many quarters are actually in the purse?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 7 **(E)** cannot be uniquely determined

Solution: (C) From the average statements it follows that 25=17+number of coins in the purse. Thus there are 8 coins actually in the purse with a value of \$ 1.28. The only way this can occur is if there are 3 pennies and thus 5 quarters.

15. For what value of the constant a do the three lines

$$2x+5y=-7 \qquad 3x-2y=18 \qquad ax+6y=2$$

all intersect at the same point?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Solution: (A) The first two lines intersect at the point $(4,-3)$. So $4a-18=2$ or $a=5$.

16. The first three figures of a certain sequence of figures are shown below on an equilateral triangle grid. Each successor figure is obtained recursively from its predeccesor by this rule: any two or more consecutive dots on a grid line generate new neighboring dots on that grid line, on either side, where no dot was before. All previous dots remain. How many dots does the 5th figure in this sequence have?

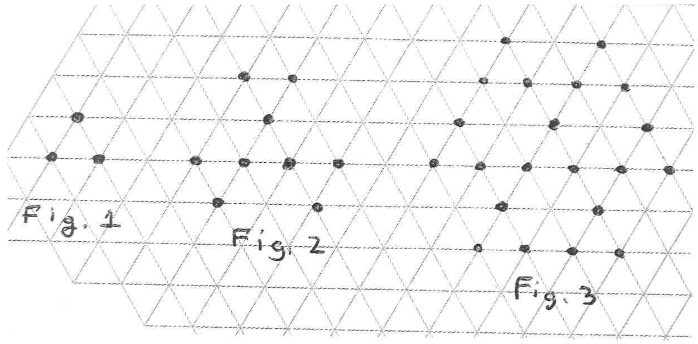
(A) 34 **(B)** 36 **(C)** 57 **(D)** 59 **(E)** 64

Solution: (C) The sequence commences: $3, 9, 21, 36, 57, 81, \ldots$.

17. If the equations $x^2+ax+21=0$ and $2x^2+19x+35=0$ have a solution in common, what could be the value of the constant a?

(A) -10 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 10

Solution: (E) The roots of the 2nd quadratic are -7 and $-5/2$. When $x=-7$ is substituted into the 1st quadratic the result is $a=10$. For $x=-5/2$ then $a=10.9$.



18. Amanda and Blake together can paint a house in 749 hours. Blake and Cathy together paint it in 535 hours. Cathy and Amanda together paint it in 642 hours. How long would it take in hours to paint the house if all three work together?

(A) 400 **(B)** 420 **(C)** 430 **(D)** 440 **(E)** 460

Solution: (B) Let A and B do the job in c hours, etc... and let r_A be the rate at which A works, etc..., then $r_A + r_B = 1/c$. etc... Adding these three equations gives: $r_A + r_B + r_C = (1/2)(1/a + 1/b + 1/c)$ so the time for all 3 together is $T = 2/(1/a + 1/b + 1/c)$ which in this case equals 420. The calculation is much facilitated by noting that the hours stated in the problem are all multiples of 107.

Spring 2013 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. Express the fraction

$$\cfrac{1}{1+\cfrac{1}{3+\cfrac{1}{5}}}$$

in lowest terms.

(A) 1/8 **(B)** 11/15 **(C)** 15/21 **(D)** 16/21 **(E)** 21/16

Solution: D First, $3 + 1/5 = 16/5$. Then $1 + 5/16 = 21/16$. So the answer is $16/21$.

2. The square root of 20000 lies between

(A) 130 and 131 **(B)** 140 and 141 **(C)** 141 and 142 **(D)** 142 and 143 **(E)** 10,000 and 10,001

Solution: C We have $\sqrt{20000} = 100\sqrt{2} \approx 141.4$.

3. How many positive factors does 2013 have?

(A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

Solution: B From $2013 = 3 \cdot 11 \cdot 61$, there are $2 \cdot 2 \cdot 2 = 8$ factors of 2013.

4. Four roses and two tulips are to be arranged in a circle. Two such arrangements are considered to be the same if and only if each can be rotated into the other. How many distinct arrangements are possible?

(A) 3 **(B)** 4 **(C)** 8 **(D)** 12 **(E)** 15

Solution: A The only possible cases are: (1) the two tulips occur together; (2) the two tulips are separated by a single rose on one side and 3 on the other; (3) the two tulips are separated by two roses on either side.

5. When a cyclist gets a puncture she has just completed three-fourths of her route. She finishes her route by walking. If she spent twice as much time walking as biking, how many times faster does she bike than walk?

(A) 4 **(B)** 4.5 **(C)** 5 **(D)** 5.5 **(E)** 6

Solution: (E) Let k be how many times faster she bikes than walks. Let w be how far she walked. Then $1/w = 2 \cdot 3/(kw)$ which reduces to $k = 6$.

6. Which of the following equations has exactly two solutions over the real numbers?

(A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
(D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

Solution: (D) Its solution set is $\{-12, 12\}$. The others do not have exactly two real solutions.

7. If the equations $x^2 + ax + 21 = 0$ and $2x^2 + 19x + 35 = 0$ have a solution in common, what could be the value of the constant a?

(A) -10 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 10

Solution: (E) The roots of the 2nd quadratic are -7 and $-5/2$. When $x = -7$ is substituted into the 1st quadratic the result is $a = 10$. For $x = -5/2$ then $a = 10.9$.

8. An off-center balance does balance when pan A has a weight of 600 grams while pan B has a weight of 900 grams. If a weight of 400 grams is added to pan A, how many grams must be added to pan B to restore the balance? Neglect the mass of the pans, beams, etc...

(A) 400 **(B)** 500 **(C)** 600 **(D)** 700 **(E)** 900

Solution: (C) We can suppose that pan A is 3 units from the balance point while pan B is 2 units from it. So $1000 \cdot 3 = (900 + x)2$ or $1200 = 2x$ or $x = 600$.

9. Twenty-seven unit cubes are assembled to form a $3 \times 3 \times 3$ cube. If two of the unit cubes are then chosen at random, what is the probability they share a face?

(A) $2/13$ **(B)** $3/11$ **(C)** $1/4$ **(D)** $3/16$ **(E)** $1/3$

Solution: (A) This probability is the sum of these probabilities: the probability that the first cube is a corner and the second shares a face with it, that the first cube is a center edge and the second share, that the first is a center face and the second shares, and that the first is the center cube and the second shares. This gives the answer of

$$(8/27)(3/26) + (12/27)(4/26) + (6/27)(5/26) + (1/27)(6/26) = 2/13$$

10. In pentagon $ABCDE$, $AB = AE = 3$, $BC = DE = 1$, $CD = 3$, $\angle B = \angle E$, and $\angle A$ is right. The area of this pentagon lies between

(A) 6 and 7 **(B)** 7 and 8 **(C)** 8 and 9
(D) 9 and 10 **(E)** 10 and 11

Solution: (B) Draw BE. Then $\triangle ABE$ is an isosceles right triangle with area $9/2$ and $BE = 3\sqrt{2}$. Isosceles trapezoid $BEDC$ has height h satisfying $h^2 = (18\sqrt{2} - 23)/4$. Using $\sqrt{2} \approx 1.4$ gives $h^2 \approx 2.2/4$ or $h \approx 1.5/2$ where we used $15^2 = 225$. Thus the area of $BEDC$ is about $(1/2)(3 + 4.2)(1.5/2)$, which is about 2.88. So the area of the pentagon is about $4.5 + 2.88 = 7.38$, very safely between 7 and 8.

11. A rectangle with area 125 has its sides in the ratio of $4:5$. What is the perimeter of this rectangle?

(A) 18 **(B)** 22.5 **(C)** 36 **(D)** 45 **(E)** 54

Solution: (D) Let the sides be $4a$ and $5a$. So the perimeter equals $18a$. But $20a^2 = 125$ or $a = 5/2$. Thus $18a = 9 \cdot 5 = 45$.

12. A solid opaque cube of side length 5 meters rests on flat ground. It is illuminated only by a powerful point-source of light located 5 meters above one of the cube's top corners. Find the area in square meters of the shadow cast by the cube on the ground.

(A) 75 **(B)** 85 **(C)** $50\sqrt{2}$ **(D)** 91 **(E)** 100

Solution: (A) The shadow together with the bottom of the cube form a square of side 10. Subtracting the bottom of the cube (because the cube is opaque) gives a shadow of area $100 - 25 = 75$.

13. For what value of the constant a do the three lines

$$2x + 5y = -7 \qquad 3x - 2y = 18 \qquad ax + 6y = 2$$

all intersect at the same point?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 8 **(E)** 9

Solution: (A) The first two lines intersect at the point $(4, -3)$. So $4a - 18 = 2$ or $a = 5$.

14. Two sides of a parallelogram lie along the lines $x - y + 1 = 0$ and $2x + 3y - 6 = 0$. If the diagonals of the parallelogram meet at the point $(1, 1/2)$, find the area of this parallelogram.

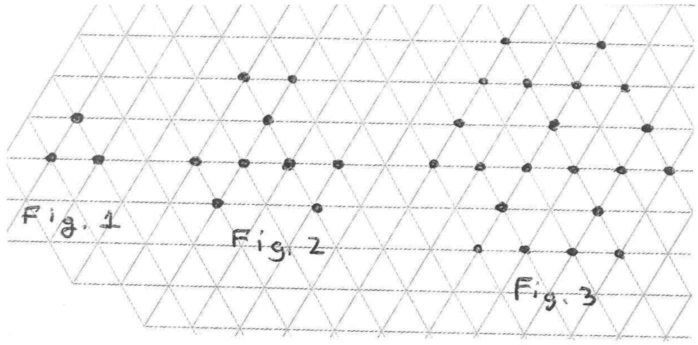
(A) $3/2$ **(B)** 2 **(C)** $5/2$ **(D)** $18/7$ **(E)** 3

Solution: (E) The two sides must be adjacent so the intersection of the given lines, which is $(3/5, 8/5)$, must be a vertex of the parallelogram. Then $(1, 1/2)$ is the midpoint between this vertex and the one opposite it, which is then easily found to be $(7/5, -3/5)$. The line through this second found vertex parallel to $2x + 3y = 6$ is $2x + 3y = 1$, which intersects $x - y = -1$ at the third found vertex $(-2/5, 3/5)$. The fourth vertex is found as was the second and is $(12/5, 2/5)$. Applying the shoelace area method to these four vertices gives an area of 3.

15. The first three figures of a certain sequence of figures are shown below on an equilateral triangle grid. Each successor figure is obtained recursively from its predeccesor by this rule: any two or more consecutive dots on a grid line generate new neighboring dots on that grid line, on either side, where no dot was before. All previous dots remain. How many dots does the 5th figure in this sequence have?

(A) 34 **(B)** 36 **(C)** 57 **(D)** 59 **(E)** 64

Solution: (C) The sequence commences: $3, 9, 21, 36, 57, 81, \ldots$.



16. Given the three points $(2013, -1863)$, $(1776, -1812)$, and $(1181, -1492)$ in the coordinate plane, a fourth point (a, b) is called a *complementing* point if it along with the given three points form the vertices of a parallelogram. Find the sum of all the coordinates of all the complementing points of the given three points.

    (A) -197 **(B)** 0 **(C)** 216 **(D)** 631 **(E)** 783

    Solution: (A) For any triangle, let its vertices V_1, V_2, and V_3, be viewed as vectors each with initial point at the origin. Then there are 3 complementing points $P_1 = V_2 + V_3 - V_1$, $P_2 = V_1 + V_3 - V_2$, and $P_3 = V_1 + V_2 - V_3$ giving the nice result that $P_1 + P_2 + P_3 = V_1 + V_2 + V_3$. Thus our answer is $2013 + 1776 + 1181 - 1863 - 1812 - 1492 = -197$.

17. Quadrilateral $PQRS$ is inscribed in a circle. Segments PQ and SR are extended to meet at T. If $\angle SPQ = 80^\circ$ and $\angle PQR = 130^\circ$, find in degrees the measure of $\angle T$.

    (A) 50 **(B)** 53 **(C)** 57 **(D)** 60 **(E)** 61

    Solution: (A) Let $\overset{\frown}{QR} = x$. Then $\overset{\frown}{SR} = 160 - x$ and consequently $\overset{\frown}{PS} = 100 + x$. But now $\angle T = (1/2)(100 + x - x) = 50$.

18. A circle of radius 9 is externally tangent to a second circle of radius b. If a common tangent to the two circles has length 12, what is the value of b?

    (A) 3.5 **(B)** 4 **(C)** 6 **(D)** 7.5 **(E)** 9

    Solution: (B) Let a and b be the radii with $a > b$. By the Pythagorean theorem, $(a + b)^2 = (a - b)^2 + 12^2$, so that $b = 36/a$. In our case, $a = 9$, so that $b = 4$.

Spring 2013 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. If 10% of a is b what is 10% of b?

   (A) $100a$ **(B)** $10a$ **(C)** a **(D)** $.1a$ **(E)** $.01a$

   Solution: (E) From $.1a = b$ we have $.1b = .01a$.

2. When a cyclist gets a puncture she has just completed three-fourths of her route. She finishes her route by walking. If she spent twice as much time walking as biking, how many times faster does she bike than walk?

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   Solution: (E) Let k be how many times faster she bikes than walks. Let w be how far she walked. Then $1/w = 2 \cdot 3/(kw)$ which reduces to $k = 6$.

3. How many seconds are there in exactly six weeks?

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   Solution: (D) Note that $6 \cdot 7 \cdot 24 \cdot 60 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 180 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 120 = 10!$

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   Solution: B Twice as many carpenters will build twice as many cabinets in the same amount of time.

5. Which of the following equations has exactly two solutions over the real numbers?

   (A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
   (D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

   Solution: (D) Its solution set is $\{-12, 12\}$. The others do not have exactly two real solutions.

6. In the following arrangement of the positive integers, in which column, counting from left to right, does 7021 appear?

$$\begin{array}{cccc} & & & \cdot \\ & & 5 & \cdots \\ & 2 & 6 & \cdots \\ 1 & 3 & 7 & \cdots \\ & 4 & 8 & \cdots \\ & & 9 & \cdots \\ & & & \cdot \end{array}$$

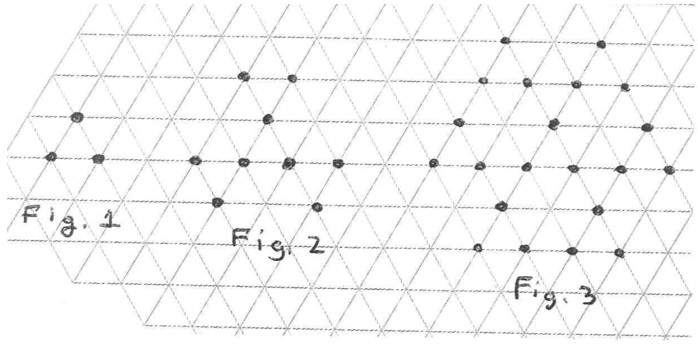
   (A) 43 **(B)** 51 **(C)** 52 **(D)** 84 **(E)** 99

   Solution: (D) Note that the last entry of each column forms the perfects squares in order. Since $83^2 < 7021 < 84^2$, 7021 occurs in the 84th column.

7. Using estimation, the number of digits in the number 2^{50} is

   (A) between 6 and 10 inclusive
   (B) between 11 and 15 inclusive
   (C) between 16 and 20 inclusive
   (D) between 21 and 25 inclusive
   (E) 26 or more

   Solution: C Since $2^{10} = 1024$ then $2^{50} > 10^{15}$ and so 2^{50} has at least 16 digits. At the same time $2^{50} < (2000)^5 = 32 * 10^{15}$ and this last number has 17 digits.



8. The first three figures of a certain sequence of figures are shown below on an equilateral triangle grid. Each successor figure is obtained recursively from its predeccesor by this rule: any two or more consecutive dots on a grid line generate new neighboring dots on that grid line, on either side, where no dot was before. All previous dots remain. How many dots does the 5th figure in this sequence have?

(A) 34 **(B)** 36 **(C)** 57 **(D)** 59 **(E)** 64

Solution: (C) The sequence commences: $3, 9, 21, 36, 57, 81, \ldots$.

9. For how many positive integers n does $n!$ end in exactly eleven zeros?

(A) 0 **(B)** 3 **(C)** 5 **(D)** 8 **(E)** 11

Solution: A When tracking these zeros note that 45! ends in 10 zeros exactly, while 50! ends in 12 zeros exactly. Thus no factorial ends in exactly 11 zeros.

10. Which of these numbers is the least?

(A) $\log_8 144$ **(B)** $\log_4 72$ **(C)** $\log_{16} 288$ **(D)** $\log_2 48$ **(E)** $\log_{32} 576$

Solution: (E) Because all the other logs are greater than two from the inequalities $8^2 < 144$, $4^2 < 72$, etc., but the last log is less than two since $32^2 > 576$.

11. Quadrilateral $PQRS$ is inscribed in a circle. Segments PQ and SR are extended to meet at T. If $\angle SPQ = 80^\circ$ and $\angle PQR = 130^\circ$, find in degrees the measure of $\angle T$.

(A) 50 **(B)** 53 **(C)** 57 **(D)** 60 **(E)** 61

Solution: (A) Let $\overset{\frown}{QR} = x$. Then $\overset{\frown}{SR} = 160 - x$ and consequently $\overset{\frown}{PS} = 100 + x$. But now $\angle T = (1/2)(100 + x - x) = 50$.

12. If the equations $x^2 + ax + 21 = 0$ and $2x^2 + 19x + 35 = 0$ have a solution in common, what could be the value of the constant a?

(A) -10 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 10

Solution: (E) The roots of the 2nd quadratic are -7 and $-5/2$. When $x = -7$ is substituted into the 1st quadratic the result is $a = 10$. For $x = -5/2$ then $a = 10.9$.

13. Which transformation never changes the median of a list of a dozen distinct positive integers?

(A) adding 6 to each number in the list
(B) adding 3 to each of the three smallest numbers in the list
(C) subtracting 4 from each of the four largest numbers in the list
(D) doubling each number in the list
(E) taking the reciprocal of each number in the list

Solution: (B) Adding 3 to each of the three smallest numbers in the list can change at most the five smallest numbers or add duplicates up to the 6th smallest number in the list, since the original list has all distinct integers. Thus the sixth and seventh cannot themseleves change under this transformation, which of course preserves the median. All the other transformations can or must change the median.

14. How many different paths are there from $(0, 0)$ to $(4, 4)$ if only these three kinds of steps may be taken: (i) a unit step to the right, (ii) a unit step up, (iii) a northeast diagonal step from point (i, j) to point $(i + 1, j + 1)$?

(A) 276 **(B)** 295 **(C)** 321 **(D)** 343 **(E)** 371

Solution: (C) On the square grid use flowing numbers. You should get along the diagonal the numbers $1, 3, 13, 63, 321$. So the answer is 321. Check out this sequence on Sloan's Online Encyclopedia of Integer sequences number A001850.

15. How many solutions in radians of $\sin 2\theta = \cos 3\theta$ lie in the interval $[0, 2\pi]$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

Solution: (E) There are 6 solutions in the interval. One way is make a careful sketch. Otherwise use trig identities and algebra. No worse than a quadratic has to be solved in that case.

16. Let $f(x) = (1/4)x^2 + bx + c$ where b and c are constants. If b and c are chosen randomly and independently from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ what is the probability that the vertex of the parabola $y = f(x)$ lies on the x-axis?

(A) 1/25 **(B)** 1/20 **(C)** 1/10 **(D)** 4/25 **(E)** 1/5

Solution: (A) The vertex lies on the x axis precisely when the discriminant vanishes, so $b^2 - c = 0$, which occurs when c is a perftect square: $c = 0, 1, 4, 9$, or four times out of 100.

17. Let a, b, and n be constants, with n a positive integer. If the first three terms of the binomial expansion of $(a + x)^n$ are, in ascending powers of x, equal to $3b + 6bx + 5bx^2$, then find the value of $a + b + n$.

(A) 48 **(B)** 64 **(C)** 96 **(D)** 128 **(E)** 252

Solution: (E) We have $a^n = 3b$, $na^{n-1}6b$, and $(n(n-1)/2)a^{n-2} = 5b$. Dividing the first two equations yields $n = 2a$ and the last two $5a = 3(n-1)$. By substitution, $5a = 6a - 3$ so that $a = 3$, $n = 6$, and $b = 243$. Thus $a + b + n = 252$.

18. When $x^{101} + x^{51} + 1$ is divided by $x^3 + 1$, what is the remainder?

(A) 0 **(B)** x **(C)** $3x^2 + 4x - 2$ **(D)** -1 **(E)** $-x^2$

Solution: E The remainder must have the form $ax^2 + bx + c$. When the roots of the divisor $x^3 + 1$ are substituted into the dividend the same value results as when these roots are substituted into the remainder. From the root $x = -1$, we obtain $a - b + c = -1$. From the root $x = 1/2 + i\sqrt{3}/2$, we obtain both $a + b = -1$ and $-a/2 + b/2 + c = 1/2$ (from real and imaginary parts respectively). The unique solution of this 3 by 3 linear system is $a = -1$, $b = 0$, and $c = 0$ so that the remainder is $-x^2$.

Spring 2013 McNabb GDCTM Contest
PreCalculus Solutions

NO Calculators Allowed

1. A rectangle with area 125 has its sides in the ratio of 4 : 5. What is the perimeter of this rectangle?

   (A) 18 **(B)** 22.5 **(C)** 36 **(D)** 45 **(E)** 54

   Solution: (D) Let the sides be $4a$ and $5a$. So the perimeter equals $18a$. But $20a^2 = 125$ or $a = 5/2$. Thus $18a = 9 \cdot 5 = 45$.

2. In the repeating decimal $0.\overline{71771}$, in which decimal place does the 2013th 7 appear?

   (A) 671st **(B)** 2014th **(C)** 2015th **(D)** 3354th **(E)** 3355th

   Solution: D 2013 divided by 3 gives 671, so the last 7 in the 671st group of 71771 is the 2013th 7. It occurs in the $670 \cdot 5 + 4$th spot, which is the 3354th spot.

3. How many seconds are there in exactly six weeks?

   (A) 7! **(B)** 8! **(C)** 9! **(D)** 10! **(E)** 12!

   Solution: (D) Note that $6 \cdot 7 \cdot 24 \cdot 60 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 180 \cdot 60 = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 120 = 10!$

4. The sum of a set of numbers is the sum of all the numbers in that set. How many subsets of the set $\{1, 2, 3, 4, 5, 6, 7\}$ have a sum of 12?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

   Solution: (E) There are exactly 8 such subsets: $\{7, 5\}$, $\{7, 4, 1\}$, $\{7, 3, 2\}$, $\{6, 5, 1\}$, $\{6, 4, 2\}$, $\{6, 1, 2, 3\}$, $\{5, 4, 3\}$, $\{5, 4, 2, 1\}$.

5. The value of
   $$1 + 2 + 3 + 4 - 5 + 6 + 7 + 8 + 9 - 10 + \cdots + 46 + 47 + 48 + 49 - 50$$
   is equal to

   (A) 600 **(B)** 650 **(C)** 725 **(D)** 750 **(E)** 800

   Solution: (C) The sum is $5 + 20 + 35 + \cdots + 140 = (145)5 = 725$.

6. A solid opaque cube of side length 5 meters rests on flat ground. It is illuminated only by a powerful point-source of light located 5 meters above one of the cube's top corners. Find the area in square meters of the shadow cast by the cube on the ground.

   (A) 75 **(B)** 85 **(C)** $50\sqrt{2}$ **(D)** 91 **(E)** 100

   Solution: (A) The shadow together with the bottom of the cube form a square of side 10. Subtracting the bottom of the cube (because the cube is opaque) gives a shadow of area $100 - 25 = 75$.

7. Let a, b, and n be constants, with n a positive integer. If the first three terms of the binomial expansion of $(a + x)^n$ are, in ascending powers of x, equal to $3b + 6bx + 5bx^2$, then find the value of $a + b + n$.

   (A) 48 **(B)** 64 **(C)** 96 **(D)** 128 **(E)** 252

   Solution: (E) We have $a^n = 3b$, $na^{n-1}6b$, and $(n(n-1)/2)a^{n-2} = 5b$. Dividing the first two equations yields $n = 2a$ and the last two $5a = 3(n - 1)$. By substitution, $5a = 6a - 3$ so that $a = 3$, $n = 6$, and $b = 243$. Thus $a + b + n = 252$.

8. Which of these numbers is the least?

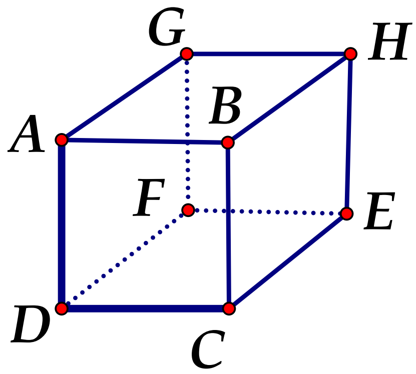
   (A) $\log_8 144$ **(B)** $\log_4 72$ **(C)** $\log_{16} 288$ **(D)** $\log_2 48$ **(E)** $\log_{32} 576$

   Solution: (E) Because all the other logs are greater than two from the inequalities $8^2 < 144$, $4^2 < 72$, etc., but the last log is less than two since $32^2 > 576$.

9. In cube $ABCDEFGH$ shown find $\cot \angle DBF$

   (A) $2/\sqrt{6}$ **(B)** $5/6$ **(C)** 1 **(D)** $\sqrt{2}$ **(E)** $6/5$

   Solution: D Let acute angle $\theta = \angle DBF$. From the Law of Cosines applied to $\triangle DBF$, $1 = 5 - 2\sqrt{2}\sqrt{3}\cos\theta$. Thus $\cos\theta = \sqrt{2}/\sqrt{3}$ and $\sin\theta = 1/\sqrt{3}$. Finally, $\cot\theta = \sqrt{2}$.



10. Which of the following equations has exactly two solutions over the real numbers?

(A) $x^2 - 6x + 9 = 0$ **(B)** $5x = 2(5 - 7x)$ **(C)** $|x + 8| = -5$
(D) $|x| = 12$ **(E)** $x^2 + 1 = 0$

Solution: (D) Its solution set is $\{-12, 12\}$. The others do not have exactly two real solutions.

11. How many solutions in radians of $\sin 2\theta = \cos 3\theta$ lie in the interval $[0, 2\pi]$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

Solution: (E) There are 6 solutions in the interval. One way is make a careful sketch. Otherwise use trig identities and algebra. No worse than a quadratic has to be solved in that case.

12. Recall that $i = \sqrt{-1}$. What is the sum of the infinite geometric series $\sum_{n=0}^{\infty}(i/2)^n$?

(A) $-\frac{1}{5} + \frac{2}{5}i$ **(B)** $\frac{3}{5} - \frac{1}{5}i$ **(C)** $\frac{4}{5} + \frac{2}{5}i$ **(D)** 0 **(E)** i

Solution: (C) $a/(1-r) = 1/(1 - i/2) = (1 + i/2)/(5/4) = (4 + 2i)/5 = 4/5 + (2/5)i$.

13. Given the three points $(2013, -1863)$, $(1776, -1812)$, and $(1181, -1492)$ in the coordinate plane, a fourth point (a, b) is called a *complementing* point if it along with the given three points form the vertices of a parallelogram. Find the sum of all the coordinates of all the complementing points of the given three points.

(A) -197 **(B)** 0 **(C)** 216 **(D)** 631 **(E)** 783

Solution: (A) For any triangle, let its vertices V_1, V_2, and V_3, be viewed as vectors each with initial point at the origin. Then there are 3 complementing points $P_1 = V_2 + V_3 - V_1$, $P_2 = V_1 + V_3 - V_2$, and $P_3 = V_1 + V_2 - V_3$ giving the nice result that $P_1 + P_2 + P_3 = V_1 + V_2 + V_3$. Thus our answer is $2013 + 1776 + 1181 - 1863 - 1812 - 1492 = -197$.

14. When $x^{101} + x^{51} + 1$ is divided by $x^3 + 1$, what is the remainder?

(A) 0 **(B)** x **(C)** $3x^2 + 4x - 2$ **(D)** -1 **(E)** $-x^2$

Solution: E The remainder must have the form $ax^2 + bx + c$. When the roots of the divisor $x^3 + 1$ are substituted into the dividend the same value results as when these roots are substituted into the remainder. From the root $x = -1$, we obtain $a - b + c = -1$. From the root $x = 1/2 + i\sqrt{3}/2$, we obtain both $a + b = -1$ and $-a/2 + b/2 + c = 1/2$ (from real and imaginary parts respectively). The unique solution of this 3 by 3 linear system is $a = -1$, $b = 0$, and $c = 0$ so that the remainder is $-x^2$.

15. Let $f(x) = (1/4)x^2 + bx + c$ where b and c are constants. If b and c are chosen randomly and independently from the set of digits $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ what is the probability that the vertex of the parabola $y = f(x)$ lies on the x-axis?

(A) $1/25$ **(B)** $1/20$ **(C)** $1/10$ **(D)** $4/25$ **(E)** $1/5$

Solution: (A) The vertex lies on the x axis precisely when the discriminant vanishes, so $b^2 - c = 0$, which occurs when c is a perftect square: $c = 0, 1, 4, 9$, or four times out of 100.

16. A careless librarian has reshelved the 5 volumes of an art encyclopedia in the correct order. Each volume has its spine facing out, which is correct of course, but has a 1/4 probability of being upside down. What is the probability that exactly one pair of front covers are now face to face?

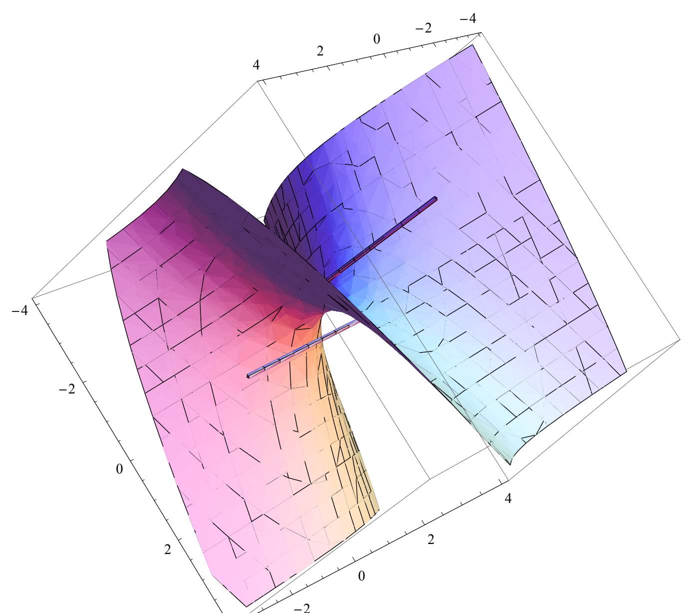
(A) 1/64 **(B)** 2/31 **(C)** 3/16 **(D)** 5/24 **(E)** 69/128

Solution: (E) By cases. Case (i). Exactly one book is placed upside down. Then there are four ways this leads to a single instance of face-to-face. Case (ii). Two books are placed upside down. This gives six instances. Case (iii). Three books are placed upside down. This gives six instances as well. Case (iv). Four books are placed upside down. This gives 4 instances. Thus our probability is

$$4(3/4)^3(1/4) + 6(3/4)^3(1/4)^2 + 6(3/4)^2(1/4)^3 + 4(3/4)(1/4)^4 = 69/128$$

17. The set of points in space equidistant from two skew lines is

(A) the empty set **(B)** a single point **(C)** a line
(D) the union of two intersecting lines **(E)** none of the above



Solution: (E) As an illustrative example to show the correct geometry take the lines to have parameterizations $(0,0,t)$ and $(-1,s,1)$. The set of points equidistant from a line is a cylinder with that line as axis. The two cylinders here for common distance r have the equations $x^2 + y^2 = r^2$ and $(x+1)^2 + (z-1)^2 = r^2$. Equating and simplifying yields $(z-1)^2 - y^2 = -2x - 1$ whose surface consists of the disjoint union of an infinite number of hyperbolas and a pair of lines.

18. In triangle ABC, the angle bisector CD of $\angle A$ has point D on side AB. If $AC = 1$, $BC = \sqrt{3}$, $AD = \sqrt{3} - 1$ and $DB = 3 - \sqrt{3}$, then what is the length CD?

(A) $\sqrt{1+\sqrt{3}}$ **(B)** $\sqrt{6-3\sqrt{3}}$ **(C)** 9/10 **(D)** 1 **(E)** $1/\sqrt{2}$

Solution: (B) Note that $\angle A = 60°$ because the sides of the triangle ABC are in the ratio $1 : \sqrt{3} : 2$. Then apply the law of cosines to deduce that $CD = \sqrt{6-3\sqrt{3}}$.

Spring 2013 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. How many positive factors does 2013 have?

   (A) 6 **(B)** 8 **(C)** 10 **(D)** 12 **(E)** 14

   Solution: B From $2013 = 3 \cdot 11 \cdot 61$, there are $2 \cdot 2 \cdot 2 = 8$ factors of 2013.

2. The value of

$$1 + 2 + 3 + 4 - 5 + 6 + 7 + 8 + 9 - 10 + \cdots + 46 + 47 + 48 + 49 - 50$$

   is equal to

   (A) 600 **(B)** 650 **(C)** 725 **(D)** 750 **(E)** 800

   Solution: (C) The sum is $5 + 20 + 35 + \cdots + 140 = (145)5 = 725$.

3. I have two numbers in mind. The first number leaves a remainder of 4159 when divided by 5153 while the second number leaves a remainder of 5149 when divided by 5153. What is the remainder when the sum of these numbers is divided by 5153?

   (A) 3135 **(B)** 3455 **(C)** 4144 **(D)** 4155 **(E)** 4344

   Solution: D The sum of the two remainders is 9308 which must be reduced by 5153 so that the true remainder of the sum will be 4155.

4. If the equations $x^2 + ax + 21 = 0$ and $2x^2 + 19x + 35 = 0$ have a solution in common, what could be the value of the constant a?

   (A) -10 **(B)** -4 **(C)** -2 **(D)** 4 **(E)** 10

   Solution: (E) The roots of the 2nd quadratic are -7 and $-5/2$. When $x = -7$ is substituted into the 1st quadratic the result is $a = 10$. For $x = -5/2$ then $a = 10.9$.

5. Which transformation never changes the median of a list of a dozen distinct positive integers?

   (A) adding 6 to each number in the list
   (B) adding 3 to each of the three smallest numbers in the list
   (C) subtracting 4 from each of the four largest numbers in the list
   (D) doubling each number in the list
   (E) taking the reciprocal of each number in the list

   Solution: (B) Adding 3 to each of the three smallest numbers in the list can change at most the five smallest numbers or add duplicates up to the 6th smallest number in the list, since the original list has all distinct integers. Thus the sixth and seventh cannot themseleves change under this transformation, which of course preserves the median. All the other transformations can or must change the median.

6. Which of these numbers is the least?

   (A) $\log_8 144$ **(B)** $\log_4 72$ **(C)** $\log_{16} 288$ **(D)** $\log_2 48$ **(E)** $\log_{32} 576$

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   (A) $1/64$ **(B)** $2/31$ **(C)** $3/16$ **(D)** $5/24$ **(E)** $69/128$

   Solution: (E) By cases. Case (i). Exactly one book is placed upside down. Then there are four ways this leads to a single instance of face-to-face. Case (ii). Two books are placed upside down. This gives six instances. Case (iii). Three books are placed upside down. This gives six instances as well. Case (iv). Four books are placed upside down. This gives 4 instances. Thus our probability is

$$4(3/4)^3(1/4) + 6(3/4)^3(1/4)^2 + 6(3/4)^2(1/4)^3 + 4(3/4)(1/4)^4 = 69/128$$

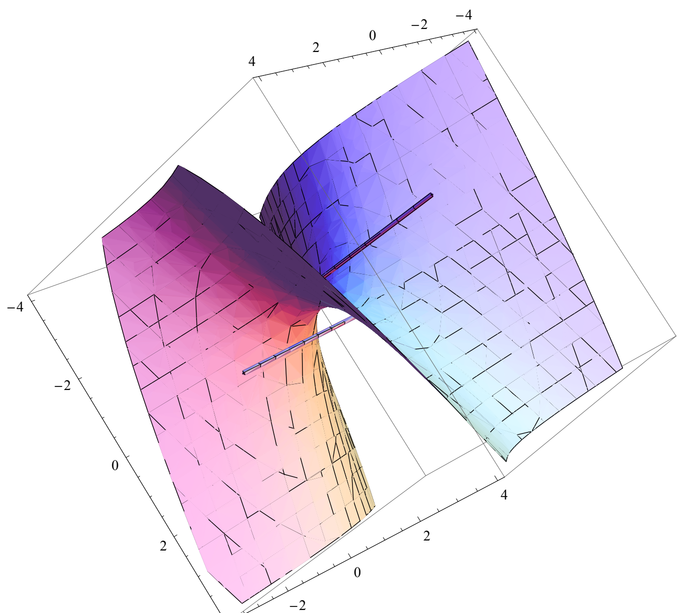
8. Recall that $i = \sqrt{-1}$. What is the sum of the infinite geometric series $\sum_{n=0}^{\infty}(i/2)^n$?

(A) $-\frac{1}{5}+\frac{2}{5}i$ **(B)** $\frac{3}{5}-\frac{1}{5}i$ **(C)** $\frac{4}{5}+\frac{2}{5}i$ **(D)** 0 **(E)** i

Solution: (C) $a/(1-r) = 1/(1-i/2) = (1+i/2)/(5/4) = (4+2i)/5 = 4/5+(2/5)i$.

9. The set of points in space equidistant from two skew lines is

(A) the empty set **(B)** a single point **(C)** a line
(D) the union of two intersecting lines **(E)** none of the above



Solution: (E) As an illustrative example to show the correct geometry take the lines to have parameterizations $(0,0,t)$ and $(-1,s,1)$. The set of points equidistant from a line is a cylinder with that line as axis. The two cylinders here for common distance r have the equations $x^2+y^2=r^2$ and $(x+1)^2+(z-1)^2=r^2$. Equating and simplifying yields $(z-1)^2-y^2=-2x-1$ whose surface consists of the disjoint union of an infinite number of hyperbolas and a pair of lines.

10. How many solutions in radians of $\sin 2\theta = \cos 3\theta$ lie in the interval $[0, 2\pi]$?

(A) 0 **(B)** 2 **(C)** 3 **(D)** 4 **(E)** 6

Solution: (E) There are 6 solutions in the interval. One way is make a careful sketch. Otherwise use trig identities and algebra. No worse than a quadratic has to be solved in that case.

11. The integral

$$\int_0^{\pi/2} \frac{1}{1+\cos\theta}\,d\theta$$

has value

(A) 3/5 **(B)** 5/6 **(C)** 1 **(D)** 7/5 **(E)** diverges

Solution: (C) Use the trig identity $1+\cos\theta = 2\cos^2(\theta/2)$ which quickly show an antiderivative is $\tan(\theta/2)$.

12. Find the minimum possible value of the expression $6\cos x + 2\cos 2x + 5$.

(A) 2/3 **(B)** 3/4 **(C)** 4/5 **(D)** 5/6 **(E)** 1

Solution: (B) Rewrite as $4\cos^2 x + 6\cos x + 3$ and view as a parabola with vertex when $\cos x = -3/4$ which of course is possible. Using this value for cosine, the minimum value of the original expression is $4(-3/4)^2+6(-3/4)+3$ or $3/4$.

13. A thin rod lies along the x-axis with endpoints at $x=2$ and $x=8$. If the density of the rod at each point is directly proportional to the point's distance to the origin, what is the x-coordinate of the center of mass of the rod?

(A) 19/5 **(B)** 4 **(C)** 14/3 **(D)** 28/5 **(E)** 5

Solution: (D) We have the center of mass $\overline{x} = \int_2^8 x^2\,dx / \int_2^8 x\,dx = (8^3-2^3)/90 = 28/5$.

14. How many values of the constant k satisfy both: (i) $k \geq 1$ and (ii) $\int_0^k (2k-2)x^k \, dx = 80$?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4

Solution: (B) Let $g(k)$ be the given integral. Then g is strictly increasing on $[1, \infty)$ because $g'(k) = (2k-2)k^k + \int_1^k 2x^k \, dx + \int_1^k (2k-2)(\ln k)x^k \, dx$ is positive on $(1, \infty)$ as it is the sum of three positive numbers there. At the same time $g(3) = 80$ so there is exactly one solution.

15. Determine

$$\lim_{n \to \infty} \int_0^{\pi/6} (\sin x)^n \, dx$$

(A) 0 **(B)** $1/10$ **(C)** $\pi/12$ **(D)** $1/2$ **(E)** does not exist

Solution: (A) On the interval of integration $(\sin x)^n \leq (1/2)^n$ so the corresponding integral is greater than or equal to zero and less than or equal to $(\pi/6)(1/2^n)$. So by the Squeeze Theorem the limit is 0.

16. The improper integral $\int_0^\infty \frac{1}{1+e^x} \, dx$ has the value

(A) $\ln 2$ **(B)** $1/2$ **(C)** $2/3$ **(D)** e **(E)** does not converge

Solution: (A) Multiply numerator and denominator of the integrand by e^{-x} then use substitution with $u = 1 + e^{-x}$. Then the improper integral becomes $\int_1^2 1/u \, du = \ln 2$.

17. Given that $\int_0^{10} \ln(x^2 - 10x + 26) \, dx = k$ then find the value of

$$\int_0^{10} x \ln(x^2 - 10x + 26) \, dx$$

(A) 0 **(B)** k **(C)** $2k$ **(D)** $k \ln 2$ **(E)** $5k$

Solution: (E) In the first given integral use the substitution $u = x - 5$ so that it equals $\int_5^5 \ln(u^2 + 1) \, du$. Use the same substitution in the second given integral so that it equals $\int_{-5}^5 u \ln(u^2+1) + 5\ln(u^2+1) \, du = 5k$ because the first term is an odd function.

18. The coefficient of x^8 in the Maclaurin power series of $f(x) = \frac{1+2x}{1-x-x^2}$ is equal to

(A) 47 **(B)** 76 **(C)** 91 **(D)** 101 **(E)** 123

Solution: (B) Let $f(x) = \sum c_n x^n$ so that $1 + 2x = (1 - x - x^2) \sum c_n x^n$. Multiply this out and match coefficients to obtain $c_0 = 1, c_1 = 3$, and $c_{n+2} = c_{n+1} + c_n$. Use this Fibonacci like sequence to obtain $c_8 = 76$.