Spring 2012 McNabb GDCTM Contest
Prealgebra Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. How many of the first 20 natural numbers, that is, the set $\{1, 2, 3, \cdots, 20\}$, are composite? Recall that 1 is neither prime nor composite.

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

   Solution: (B) The eleven composites are: 4,6,8,9,10,12,14,15,16,18, and 20.

2. The value of $-3 - 3^2 - 3^3$ is equal to

   (A) -41 **(B)** -40 **(C)** -39 **(D)** -37 **(E)** -36

   Solution: (C) $-3 - 9 - 27 = -12 - 27 = -39$.

3. A clever saleswoman is counting out envelopes for a customer. Every package of envelopes contains 80 envelopes. The saleswoman can count out 8 envelopes in 8 seconds. How many seconds does she need to count out 56 envelopes?

   (A) 24 **(B)** 48 **(C)** 56 **(D)** 72 **(E)** 80

   Solution: (A) She is clever so she counts the complement! $80 - 56 = 24$.

4. A jar contains only blue and green marbles in the ratio of 3 blue for every 5 green. If 3 green marbles are removed and replaced by 3 blue marbles, the ratio of blue to green marbles becomes one to one. How many green marbles were in the jar originally?

   (A) 15 **(B)** 18 **(C)** 21 **(D)** 24 **(E)** 27

   Solution: (A) Let $3a$ and $5a$ be the number of blue and green marbles present originally. Then $5a - 3 = 3a + 3$ means $a = 3$. So $5a = 15$ green marbles were there originally.

5. What is the largest possible product of two positive odd integers whose sum is 40?

   (A) 39 **(B)** 279 **(C)** 300 **(D)** 399 **(E)** 400

   Solution: (D) Make the factors as close together as possible, so choose 19 and 21. Note $19 \cdot 21 = 399$.

6. What is the remainder when the sum

   $$1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 + 7^3 + 8^3$$

   is divided by 9?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

   Solution: (A) Since 8 and -1 both have the same remainder when divided by 9, then 8^3 and $(-1)^3$ also have the same remainder when divided by 9. But $1^3 + (-1)^3 = 0$. This show that when $1^3 + 8^3$ is divided by 9 the remainder must be zero. The same works for the other remaining pairs, such as $2^3 + 7^3$, etc.. Thus the whole sum gives a remainder of zero when divided by 9.

7. Before a grape is dried to become a raisin it is 94% water, while a raisin is only 25% water. If only water is evaporated from the grapes, how many kilograms of raisins can be made from 60 kilograms of grapes?

(A) 4.8 **(B)** 18.2 **(C)** 24.6 **(D)** 27.6 **(E)** 45

Solution: (A) The dry matter of the grapes is $(0.06)60 = 3.6$kg. This constitutes $3/4$ of the mass of the raisins, so the mass of the raisins is $4/3$ of 3.6, which comes to 4.8.

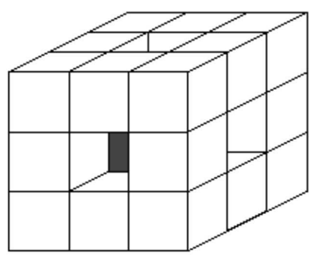
8. Mr. and Mrs. Reynolds have three daughters and three sons. At Easter each member of the family buys one chocolate Easter egg for everybody else in the family. How many Easter eggs will the Reynolds family buy in total?

(A) 28 **(B)** 32 **(C)** 40 **(D)** 56 **(E)** 64

Solution: (D) Each family member buys 7 eggs, so eight times seven, or 56, eggs were bought in total.

9. Twenty seven small $1 \times 1 \times 1$ cubes are glued together to form a $3 \times 3 \times 3$ cube. Then the center small cube and the small cubes at the center of each face are removed. What is the surface area of the resulting solid?

(A) 56 **(B)** 64 **(C)** 72 **(D)** 84 **(E)** 96



Solution: (C) One way to count is to note that there are 8 corner cubes, showing 3 faces apiece and there are are 12 middle pieces showing 4 faces apiece. Thus the surface of the solid measures $8 \cdot 3 + 12 \cdot 4 = 24 + 48 = 72$.

10. A train 2700 meters long passes a signal in 135 seconds. At that rate, how many seconds does it take to cross a bridge 1000 meters in length?

(A) 50 **(B)** 110 **(C)** 135
(D) 160 **(E)** 185

Solution: (E) The velocity of the train is $2700/135 = 20$ meters per second. So the nose of the train takes 50 seconds to cross to the other side, but then the rest of the train takes another 135 seconds to get all across. So altogether the train takes $50 + 135 = 185$ seconds to cross.

11. Which integer below cannot be written as the sum of the squares of two integers?

(A) 289 **(B)** 353 **(C)** 450 **(D)** 481 **(E)** 503

Solution: (E) All the rest can be. For example, $289 = 17^2 + 0^2$, $353 = 289 + 64 = 17^2 + 8^2$, $450 = 225 + 225 = 15^2 + 15^2$, and $481 = 20^2 + 9^2$. That 503 cannot be so written can be shown be inspection. It is also known from number theory that any integer of the form $4n + 3$, three more than a multiple of four, cannot be written as the sum of two squares.

12. The digits 7,5,3,2, and 0 are each used once to make the smallest possible 5 digit number divisible by 11. What is the hundred's digit of this number?

    (A) 0 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 7

    Solution: (D) Try the first two digits as 2 and 0. Then if the hundreds is 3, one cannot make the test for divisibility by 11 work. E.g., 2-0+3-7+5=3 and 3 is not divisible by 11. However, if the hundreds is next smallest available, that is 5, it can work as 2-0+5-3+7=11. But if the 7 and 3 are switched here it does not work. Thus 20537 is the smallest such integer.

13. A cylindrical can of juice holds 12 ounces. If the diameter of the can were to be doubled and the height halved, how many ounces would the new can hold?

    (A) 6 **(B)** 12 **(C)** 24 **(D)** 48 **(E)** 96

    Solution: (C) Doubling first the diameter makes a can with 4 times the original volume. Halving the height then cuts this to just twice the original volume.

14. If the area of a circle is increased by 300% by what percent is the circumference increased?

    (A) 50% **(B)** 100% **(C)** 150% **(D)** 200% **(E)** 300%

    Solution: (B) So the area was quadrupled which means the radius, and hence, circumference were doubled.

15. Two drovers A and B went to market with cattle. A sold 50 and then had left as many as B, who had not sold any yet. Then B sold 54 and had remaining half as many as A. How many cattle total did they have between them on their way to market?

    (A) 104 **(B)** 108 **(C)** 148 **(D)** 158 **(E)** 266

    Solution: (E) Two equations result: $A - 50 = B$ and then $B - 54 = (1/2)(A - 50)$. So $B - 54 = (1/2)B$ or $(1/2)B = 54$ or $B = 108$. Thus $A = 158$ and $A + B = 158 + 108 = 266$.

16. A cage contains birds and rabbits. There are seventeen heads and forty feet. How many rabbits are in the cage?

    (A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** 15

    Solution: (A) All birds would mean 34 feet. The extra 6 feet come from 3 rabbits!

17. Water flows continuously into a 200ℓ tank at the rate of $3\ell/\mathrm{min}$ and flows continuously out at the rate of $2\ell/\mathrm{min}$. If the tank is initially 25% full, how many minutes will it take for the tank to fill completely?

    (A) 100 **(B)** 150 **(C)** 175 **(D)** 200 **(E)** 225

    Solution: (B) There is a net gain of 1 liter of water per minute. Since 75% of 200 is 150, these 150 liters take 150 minutes.,

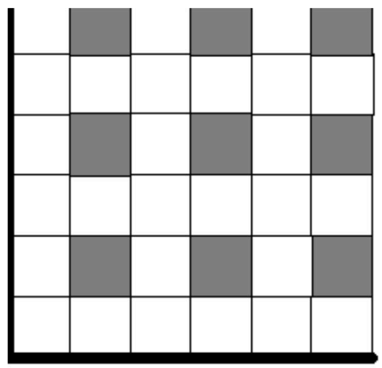
18. Jane took 5 tests, each time receiving a different score. These scores were all integers less than or equal to 100 and greater than or equal to zero. The average of her three lowest scoring tests was 84 while the average of her three highest scoring tests was 89. What is the maximum possible score of her highest scoring test?

(A) 95 **(B)** 96 **(C)** 97 **(D)** 98 **(E)** 99

Solution: (B) To maximize the highest score it is necessary to minimize the third highest score, which is done by choosing the three lowest scores to be 83, 84, and 85. Then the third and fourth highest will be 85 and 86. These two scores are together 7 points below the average of 89. So the highest could be at most 7 points above 89, that is 96.

19. A rectangular lobby is to be tiled in this pattern in such a way that the border tiles along all the edges of the lobby are white. If the lobby measures 201 tiles by 101 tiles, how many shaded tiles are required?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100



Solution: (C) The shaded tiles will be 50 rows of 100 tiles each.

20. The value of $\left(\sqrt{12}+\sqrt{3}\right)^2$ is

(A) 40 **(B)** 36 **(C)** 27 **(D)** 21 **(E)** 15

Solution: (C) Simplify to get $(2\sqrt{3}+\sqrt{3})^2 = (3\sqrt{3})^2 = 9 \cdot 3 = 27$.

21. Quadrilateral $ABCD$ has vertices in the coordinate plane as follows:
$A = (0,0), B = (5,12), C = (-3,-3)$, and $D = (0,-7)$. The perimeter of this polygon equals

(A) 28 **(B)** 35 **(C)** 42 **(D)** 45 **(E)** 46

Solution: (C) Segment AB is the hypotenuse of a 5-12-13 pythagorean triple, so $AB = 13$. Likewise segment BC is the hypotenuse of a 8-15-17 triple,so $BC = 17$. And segment CD is the hypotenuse of a 3-4-5 triple so $CD = 5$. Finally $DA = 7$. So the perimeter of the quadrilateral equals $13+17+5+7 = 42$ units.

22. Let

$$S = \frac{1+2+4+8+16}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$$

Then S equals

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

Solution: (E) Notice that 16 times the denominator equals the numerator!

23. Three pairs of husbands and wives are to be seated at a bolted-down picnic table which seats exactly six people, three to a side. If no husband is to sit on the same side as his wife, and no wife is to sit directly across from her husband, in how many ways can these six persons be seated?

(A) 24 **(B)** 60 **(C)** 64 **(D)** 80 **(E)** 96

Solution: (E) Suppose all the wives are on one side. There are two ways to pick which side and then six ways to permute the wives on that side. So 12 ways so far. But then across from one of the wives, only two of the men can sit. Once that is decided the rest is forced, so in this case there are 24 ways. The only other case is two wives on one side and the third on the other side. There are two sides to choose for the two wives to sit on, three ways to choose which two wives, and then 6 ways to seat the two wives on that side. So thirty six ways so far. Note the third wife cannot sit across from the currently empty seat because that would be where her husband would have to sit since the husbands of the two wives together would not be able to sit there either. Either way then this cannot occur. Thus the third wife must sit across from one of the other women, giving two choices. Once this choice is made, the rest is forced. Thus there are 36 times 2 or 72 ways in the second case. Thus 72+24 or 96 ways all told. Note with no restrictions there would have been six factorial, or 720 ways.

24. For how many positive integers n is the least common multiple of 30, 24, and n equal to 600?

(A) 8 **(B)** 9 **(C)** 10 **(D)** 12 **(E)** 18

Solution: (A) Take the prime factorizations: $30 = 2 \cdot 3 \cdot 5$, $24 = 2^3 \cdot 3$, and $600 = 2^3 \cdot 3 \cdot 5^2$. So n must have a 5^2 in its prime factorization, but can have 2^a where a could be 0,1,2 or 3 (giving 4 choices) and 3^b where b could be 0 or 1 (giving 2 choices). Thus there are 4 times 2 or 8 choices altogether.

25. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

(A) 5/12 **(B)** 1/11 **(C)** 1/24 **(D)** 1/22 **(E)** 1/48

Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to 1/22.

Spring 2012 McNabb GDCTM Contest
Algebra One Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. The value of $-3-3^2-3^3$ is equal to

   (A) -40 **(B)** -39 **(C)** -37 **(D)** -36 **(E)** -33

   Solution: (B) $-3-9-27=-12-27=-39$.

2. A clever saleswoman is counting out envelopes for a customer. Every package of envelopes contains 80 envelopes. The saleswoman can count out 8 envelopes in 8 seconds. How many seconds does she need to count out 56 envelopes?

   (A) 24 **(B)** 48 **(C)** 56 **(D)** 72 **(E)** 80

   Solution: (A) She is clever so she counts the complement! $80-56=24$.

3. If $a \diamond b$ equals the lesser of $1/a$ and $1/b$, find the value of $-3 \diamond (-2 \diamond (-1/2))$.

   (A) -3 **(B)** -1/2 **(C)** -1/3 **(D)** -2 **(E)** -1

   Solution: (B) $-3 \diamond (-2 \diamond (-1/2)) = -3 \diamond -2 = -1/2$.

4. What is the largest possible product of two positive odd integers whose sum is 40?

   (A) 39 **(B)** 279 **(C)** 300 **(D)** 399 **(E)** 400

   Solution: (D) Make the factors as close together as possible, so choose 19 and 21. Note $19 \cdot 21 = 399$.

5. Let
$$S = \frac{1+2+4+8+16}{1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}}$$
   Then S equals

   (A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

   Solution: (E) Notice that 16 times the denominator equals the numerator!

6. A cage contains birds and rabbits. There are seventeen heads and forty feet. How many rabbits are in the cage?

   (A) 0 **(B)** 3 **(C)** 6 **(D)** 9 **(E)** 12

   Solution: (B) All birds would mean 34 feet. The extra 6 feet come from 3 rabbits!

7. Mr. and Mrs. Reynolds have three daughters and three sons. At Easter each member of the family buys one chocolate Easter egg for everyone else in the family. How many Easter eggs will the Reynolds family buy in total?

   (A) 28 **(B)** 32 **(C)** 40 **(D)** 56 **(E)** 64

   Solution: (D) Each family member buys 7 eggs, so eight times seven, or 56, eggs were bought in total.

8. Three pairs of husbands and wives are to be seated at a bolted-down picnic table which seats exactly six people, three to a side. If no husband is to sit on the same side as his wife, and no wife is to sit directly across from her husband, in how many ways can these six persons be seated?

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10. Twenty seven small $1 \times 1 \times 1$ cubes are glued together to form a $3 \times 3 \times 3$ cube. Then the center small cube and the small cubes at the center of each face are removed. What is the surface area of the resulting solid?

    (A) 56 **(B)** 64 **(C)** 72 **(D)** 84 **(E)** 96

    Solution: (C) One way to count is to note that there are 8 corner cubes, showing 3 faces apiece and there are are 12 middle pieces showing 4 faces apiece. Thus the surface of the solid measures $8 \cdot 3 + 12 \cdot 4 = 24 + 48 = 72$.

11. Two drovers A and B went to market with cattle. A sold 50 and then had left as many as B, who had not sold any yet. Then B sold 54 and had remaining half as many as A. How many cattle total did they have between them on their way to market?

    (A) 104 **(B)** 108 **(C)** 148 **(D)** 158 **(E)** 266

    Solution: (E) Two equations result: $A - 50 = B$ and then $B - 54 = (1/2)(A - 50)$. So $B - 54 = (1/2)B$ or $(1/2)B = 54$ or $B = 108$. Thus $A = 158$ and $A + B = 158 + 108 = 266$.

12. The value of $\left(\sqrt{12}+\sqrt{3}\right)^2$ is

(A) 40 **(B)** 36 **(C)** 30 **(D)** 27 **(E)** 24

Solution: (D) Simplify to get $(2\sqrt{3}+\sqrt{3})^2 = (3\sqrt{3})^2 = 9 \cdot 3 = 27$.

13. Quadrilateral $ABCD$ has vertices in the coordinate plane as follows:
$A = (0,0), B = (5,12), C = (-3,-3)$, and $D = (0,-7)$. The perimeter of this polygon equals

(A) 28 **(B)** 35 **(C)** 42 **(D)** 45 **(E)** 46

Solution: (C) Segment AB is the hypotenuse of a 5-12-13 pythagorean triple, so $AB = 13$. Likewise segment BC is the hypotenuse of a 8-15-17 triple,so $BC = 17$. And segment CD is the hypotenuse of a 3-4-5 triple so $CD = 5$. Finally $DA = 7$. So the perimeter of the quadrilateral equals $13+17+5+7 = 42$ units.

14. What is the remainder when the sum

$$1^{111}+2^{111}+3^{111}+4^{111}+5^{111}+6^{111}+7^{111}+8^{111}+9^{111}+10^{111}$$

is divided by 11?

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Solution: (A) Since 10 has the same remainder as -1 when divided by 11, then 10^{111} has the same remainder as $(-1)^{111}$ when divided by 11. So $1^{111}+10^{111}$ has the same remainder as $1^{111}+(-1)^{111}$ or 0 when divided by 11, that is 0. The same works on the other opposite pairs, so the overall remainder is 0 as well.

15. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

(A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120

Solution: (A) First count all paths, including through the center of the cube. These paths can be labeled with 2 U's, 2 L's, and 2 B's, for up, left, and back moves. The number of 6 letter words of this type is $6!/(2!2!2!) = 720/8 = 90$. Second we count the number of paths through the center. There are 6 ways to get to the center, then 6 ways from the center to the opposite corner, so $6^2 = 36$ paths through the center. Finally, our answer is $90 - 36 = 54$.

16. Given that $x \neq 0$, $y \neq 0$, and

$$x = \frac{6}{y} - \frac{9}{xy^2}$$

what is the value of xy?

(A) 3 **(B)** 6 **(C)** 9 **(D)** 12 **(E)** cannot be uniquely determined

Solution: (A) Multiply both sides by xy^2 and gather all terms to the left to obtain $x^2y^2 - 6xy + 9$ which factors as $(xy-3)^2$, so that $xy = 3$.

17. A rectangular lobby is to be tiled in this pattern in such a way that the border tiles along all the edges of the lobby are white. If the lobby measures 201 tiles by 101 tiles, how many shaded tiles are required?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100

Solution: (C) The shaded tiles will be 50 rows of 100 tiles each.

18. Jane took 5 tests, each time receiving a different score. These scores were all integers less than or equal to 100 and greater than or equal to zero. The average of her three lowest scoring tests was 84 while the average of her three highest scoring tests was 89. What is the maximum possible score of her highest scoring test?

(A) 94 **(B)** 95 **(C)** 96 **(D)** 97 **(E)** 98

Solution: (C) To maximize the highest score it is necessary to minimize the third highest score, which is done by choosing the three lowest scores to be 83, 84, and 85. Then the third and fourth highest will be 85 and 86. These two scores are together 7 points below the average of 89. So the highest could be at most 7 points above 89, that is 96.

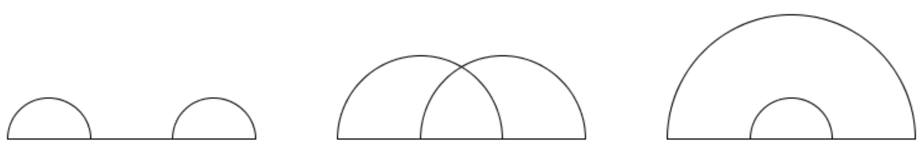
19. In rectangle $ABCD$ point P is located on side CD, closer to C than D, in such a way that $\angle APB$ is right. If $AB = 5$ and $AD = 2$, find the length of segment CP.

(A) 5/7 **(B)** 6/7 **(C)** 1 **(D)** 2 **(E)** 3

Solution: (C) Let $x = CP$. By the Pythagorean Theorem applied thrice, $x^2 + 2^2 + (5-x)^2 + 2^2 = 5^2$ or $2x^2 - 10x + 8 = 0$ or $x^2 - 5x + 4 = (x-1)(x-4) = 0$, so $x = 1$ or 4, but P is closer to C so $x = 1 = CP$ is correct.

20. Distribute 14 points along a line segment. How many distinct ways are there for pairing these points using semicircles? The case of four points is pictured above.

(A) 10395 **(B)** 40320 **(C)** 60125 **(D)** 101245 **(E)** 135135



Solution: (E) Pick one point. There are 13 choices with whom to match it. For any of the remaining 12 points there are 11 choices with whom to match it. Continue in this manner, so we see the total number of ways is $13 \cdot 11\dot{9} \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 135135$.

21. Three dice are rolled and it is known that the sum is a multiple of three. What is the probability that the sum is nine?

(A) 1/3 **(B)** 25/72 **(C)** 13/48 **(D)** 1/2 **(E)** 11/18

Solution: (B) Here is a solution, albeit inelegant. There are 1, 10, and 25 ways to respectively roll a 3, 6, and a 9, done by tables. Now there is a symmetry to take advantage of here, that there are as many ways to roll an n as there are to roll a $21 - n$. The mapping is $k \to 7 - k$. So there are 1, 10, and 25 ways to roll an 18, 15, and 12 respectively. So the desired probability is 25/72.

22. For how many integer values of k can the polynomial $12x^2 + kx + 12$ be factored as $(ax+b)(cx+d)$ where a, b, c, and d are integers with $a \neq 0$ and $b \neq 0$?

(A) 16 **(B)** 18 **(C)** 20 **(D)** 22 **(E)** 24

Solution: (A) With the AC method: $AC = 12 \cdot 12 = 144$. There are eight ways to factor 144 using two positive integers starting from $1 \cdot 144$ and ending with $12 \cdot 12$. These give the 8 possible positive k's. The other eight come from making both factors negative. So 16 k's altogether.

23. Find the coefficient of x in the expansion of

$$(x-2012)(x-2011)(x-2010)\cdots(x+2010)(x+2011)(x+2012)$$

(A) $-(2012!)^2$ **(B)** $-(1006!)^2$ **(C)** 0 **(D)** $(1006!)^2$ **(E)** $(2012!)^2$

Solution: (E) Note that the middle factor is just $(x-0) = x$. Then use difference of squares to multiply opposite pairs such as $(x-2012)(x+2012) = x^2 - 2012^2$. So the coefficient of x in the final product will be $(-2012^2)(-2011^2)\cdots(-2^2)(-1^2) = (2012!)^2$ as there are an even number of opposite signs.

24. If $16^N = 2^1 \cdot 2^3 \cdot 2^6 \cdot 2^{10} \cdots 2^k$, where the exponents follow the triangular numbers, and k is the 20th triangular number, what is the value of N?

(A) 190 **(B)** 215 **(C)** 300 **(D)** 385 **(E)** 420

Solution: (D) Use the hockey-stick identity from Pascals triangle to realize that the the sum of the first n triangular number is $\binom{n+2}{3}$, in our case $\binom{22}{3}$. Divide this number by 4 to convert to base 16 to get the correct exponent of 385.

25. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

(A) $5/12$ **(B)** $1/11$ **(C)** $1/24$ **(D)** $1/22$ **(E)** $1/48$

Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to $1/22$.

Spring 2012 McNabb GDCTM Contest
Geometry Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. A cage contains birds and rabbits. There are seventeen heads and forty feet. How many rabbits are in the cage?

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: (A) All birds would mean 34 feet. The extra 6 feet come from 3 rabbits!

2. Mr. and Mrs. Reynolds have three daughters and three sons. At Easter each member of the family buys one chocolate Easter egg for everyone else in the family. How many Easter eggs will the Reynolds family buy in total?

   (A) 28 **(B)** 32 **(C)** 40 **(D)** 56 **(E)** 64

   Solution: (D) Each family member buys 7 eggs, so eight times seven, or 56, eggs were bought in total.

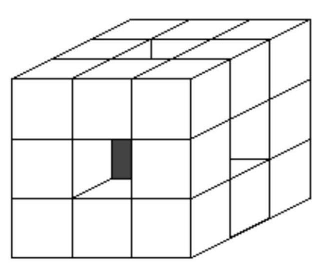
3. Quadrilateral $ABCD$ has vertices in the coordinate plane as follows:
   $A = (0, 0), B = (5, 12), C = (-3, -3)$, and $D = (0, -7)$. The perimeter of this polygon equals

   (A) 40 **(B)** 42 **(C)** 45 **(D)** 46 **(E)** 48

   Solution: (B) Segment AB is the hypotenuse of a 5-12-13 pythagorean triple, so $AB = 13$. Likewise segment BC is the hypotenuse of a 8-15-17 triple,so $BC = 17$. And segment CD is the hypotenuse of a 3-4-5 triple so $CD = 5$. Finally $DA = 7$. So the perimeter of the quadrilateral equals $13 + 17 + 5 + 7 = 42$ units.

4. Twenty seven small $1 \times 1 \times 1$ cubes are glued together to form a $3 \times 3 \times 3$ cube. Then the center small cube and the small cubes at the center of each face are removed. What is the surface area of the resulting solid?

   (A) 56 **(B)** 64 **(C)** 72 **(D)** 84 **(E)** 96

   Solution: (C) One way to count is to note that there are 8 corner cubes, showing 3 faces apiece and there are are 12 middle pieces showing 4 faces apiece. Thus the surface of the solid measures $8 \cdot 3 + 12 \cdot 4 = 24 + 48 = 72$.

5. What is the y-intercept of the perpendicular bisector of the segment with endpoints $(-2,8)$ and $(8,4)$?

   (A) -1/2 **(B)** 1/2 **(C)** 0 **(D)** -3/2 **(E)** 3/2

   Solution: (D) The slope of the given segment is $(8-4)/(-2-10) = 4/-10 = -2/5$, so the slope perpendicular is $5/2$. The midpoint has coordinates the averages of the endpoints, so $(3,6)$. The point-slope form of the equation of the perpendicular bisector is then $y-6 = (5/2)(x-3)$. Substitute $x = 0$ to the find the y-intercept at $-15/2+6 = -3/2$.

6. Three pairs of husbands and wives are to be seated at a bolted-down picnic table which seats exactly six people, three to a side. If no husband is to sit on the same side as his wife, and no wife is to sit directly across from her husband, in how many ways can these six persons be seated?

   (A) 24 **(B)** 60 **(C)** 64 **(D)** 80 **(E)** 96

   Solution: (E) Suppose all the wives are on one side. There are two ways to pick which side and then six ways to permute the wives on that side. So 12 ways so far. But then across from one of the wives, only two of the men can sit. Once that is decided the rest is forced, so in this case there are 24 ways. The only other case is two wives on one side and the third on the other side. There are two sides to choose for the two wives to sit on, three ways to choose which two wives, and then 6 ways to seat the two wives on that side. So thirty six ways so far. Note the third wife cannot sit across from the currently empty seat because that would be where her husband would have to sit since the husbands of the two wives together would not be able to sit there either. Either way then this cannot occur. Thus the third wife must sit across from one of the other women, giving two choices. Once this choice is made, the rest is forced. Thus there are 36 times 2 or 72 ways in the second case. Thus 72+24 or 96 ways all told. Note with no restrictions there would have been six factorial, or 720 ways.

7. What is the area of a triangle whose sides measure 13, 14, and 15?

   (A) 64 **(B)** 84 **(C)** 90 **(D)** 92 **(E)** 96

   Solution: (B) Drop the altitude to the side of 14. This splits the given triangle into two right triangles, one a 5-12-13, the other a 9-12-15. So that altitude measures 12, and the area of the given triangle is $(1/2)(12)(14) = 6(14) = 84$.

8. In rectangle $ABCD$ point P is located on side CD, closer to C than D, in such a way that $\angle APB$ is right. If $AB = 5$ and $AD = 2$, find the length of segment CP.

   (A) 1 **(B)** 6/5 **(C)** 5/4 **(D)** $\sqrt{2}$ **(E)** 2

   Solution: (A) Let $x = CP$. By the Pythagorean Theorem applied thrice, $x^2+2^2+(5-x)^2+2^2 = 5^2$ or $2x^2-10x+8 = 0$ or $x^2-5x+4 = (x-1)(x-4) = 0$, so $x = 1$ or 4, but P is closer to C so $x = 1 = CP$ is correct.

9. Two drovers A and B went to market with cattle. A sold 50 and then had left as many as B, who had not sold any yet. Then B sold 54 and had remaining half as many as A. How many cattle total did they have between them on their way to market?

   (A) 104 **(B)** 108 **(C)** 148 **(D)** 158 **(E)** 266

   Solution: (E) Two equations result: $A-50 = B$ and then $B-54 = (1/2)(A-50)$. So $B-54 = (1/2)B$ or $(1/2)B = 54$ or $B = 108$. Thus $A = 158$ and $A+B = 158+108 = 266$.

10. In $\triangle ABC$ points D, E, and F lie on segments $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively, in such a way that the proportions $BD/DC = 7/3$, $CE/EA = 3/2$, and $AF/FB = 4/1$ hold. If AD and FE intersect at G, what is the ratio AG/GD?

    (A) 5/6 **(B)** 6/7 **(C)** 7/8 **(D)** 8/9 **(E)** 1/1

    Solution: (D) Put a mass of 3 at B, a mass of 7 at C, and split masses of $3/4$ and $21/2$ at A to handle the transversal EF. The center of mass is then G, and will still be G when the mass of 10 is concentrated at D and the mass of $3/4 + 21/2 = 45/4$ at A. Then by the lever principle, $AG \cdot 45/4 = GD \cdot 10$, so that $AG/GD = 10 \cdot 4/45 = 40/45 = 8/9$.

11. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

    (A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120

    Solution: (A) First count all paths, including through the center of the cube. These paths can be labeled with 2 U's, 2 L's, and 2 B's, for up, left, and back moves. The number of 6 letter words of this type is $6!/(2!2!2!) = 720/8 = 90$. Second we count the number of paths through the center. There are 6 ways to get to the center, then 6 ways from the center to the opposite corner, so $6^2 = 36$ paths through the center. Finally, our answer is $90 - 36 = 54$.

12. Hezy and Zeke are both painters. Working alone Hezy can paint a certain room in 4 hours. Working alone Zeke can paint the same room in 6 hours. Let r_H, r_Z, and r_{HZ} be respectively the rates at which Hezy works alone, Zeke works alone, and Hezy and Zeke work together. Suppose for some coefficient of efficiency k satisfying $0 \le k \le 1$, these rates are related by the formula

    $$r_{HZ} = k(r_H + r_Z)$$

    Very often $k < 1$ since the two working at the same time interfere with each other to some extent. What is the value of k if in fact it takes Hezy and Zeke working together 2.7 hours to paint this room?

    (A) 6/7 **(B)** 7/8 **(C)** 8/9 **(D)** 9/10 **(E)** 10/11

    Solution: (C) Use rate times time equals distance, to get $k(1/4 + 1/6) \cdot (2.7) = 1$ so $k = 12/(5 \cdot 2.7) = 120/(5 \cdot 27) = 24/27 = 8/9$.

13. In $\triangle ABC$, the medians AD and BE are perpendicular. If $AC = 8$ and $BC = 12$, what is the length of AB?

    (A) 6 **(B)** $4\sqrt{13/5}$ **(C)** 9 **(D)** $4\sqrt{3}$ **(E)** 7

    Solution: (B) Let the medians meet at centroid G. The medians split each other in a two to one ratio, so if $GD = x$ it follows that $GA = 2x$ and if $GE = y$ then $GB = 2y$. From the Pythagorean theorem, we have the system $x^2 + 4y^2 = 36$ and $4x^2 + y^2 = 16$. Adding we have $5x^2 + 5y^2 = 52$. But $AB^2 = 4x^2 + 4y^2 = (4/5)(52) = 16(13/5)$, so $AB = 4\sqrt{13/5}$.

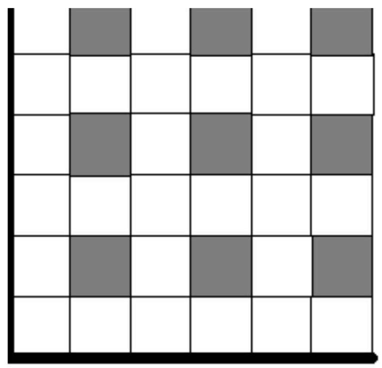
14. Consider the lines $y = 0, y = \sqrt{3}$, and $y = \sqrt{3}x$. Let C be the center of the circle that is both tangent to all three of these lines and whose x-coordinate is negative. The sum of the coordinates for the center of C can written in the form $a + b\sqrt{3}$ where a and b are rational numbers. Determine $a + b$.

    (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

Solution: (B) The center of the circle lies on the angle bisector of the angle formed by the lines $y = \sqrt{3}$ and $y = \sqrt{3}x$ and on the angle bisector of the angle formed by the lines $y = 0$ and $y = \sqrt{3}x$. Using 30-60-90 triangles the equation of the former is $y - \sqrt{3} = (1/\sqrt{3})(x - 1)$ and the latter $y = -\sqrt{3}x$. These two lines intersect at the center $(-1/2, \sqrt{3}/2)$. Thus $a + b = -1/2 + 1/2 = 0$.

15. A rectangular lobby is to be tiled in this pattern in such a way that the border tiles along all the edges of the lobby are white. If the lobby measures 201 tiles by 101 tiles, how many shaded tiles are required?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100



Solution: (C) The shaded tiles will be 50 rows of 100 tiles each.

16. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

(A) $5/12$ **(B)** $1/11$ **(C)** $1/24$ **(D)** $1/22$ **(E)** $1/48$

Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to $1/22$.

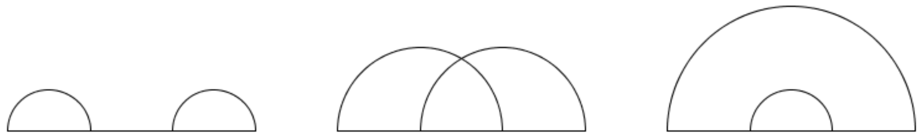
17. Let points A, B, C, and P lie in the $x - y$ plane. The coordinates of A, B, and C are $(0,0)$, $(3,0)$, and $(9,0)$ respectively. If $AP = 7$ and $BP = 6$, what is the value of CP?

(A) 8 **(B)** $\sqrt{65}$ **(C)** $25/3$ **(D)** 9 **(E)** $\sqrt{85}$

Solution: (A) Note that the area of $\triangle PBC$ has twice the area of $\triangle PBA$. Let $CP = 2x$. By Heron's formula (squared) applied to each of these triangles, $4(8 \cdot 1 \cdot 2 \cdot 5) = (6 + x)(6 - x)xx$ or $320 = 36x^2 - x^4$ or $x^4 - 36x^2 + 320 = (x^2 - 16)(x^2 - 20) = 0$. As $x > 0$ we have only possibilities: $x = 4$ or $x = \sqrt{20}$. Note that $\triangle PBA$ has an obtuse angle $\angle ABP$, so that $\angle PBC$ must be acute which in either case would force $\triangle PBC$ to be acute. Since $x = \sqrt{20}$ would force $\triangle PBC$ to be obtuse we reject that choice, and so $x = 4$ making $CP = 2x = 8$. Note this problem may also be solved by Stewart's Theorem.

18. Distribute 14 points along a line segment. How many distinct ways are there for pairing these points using semicircles? The case of four points is pictured below.

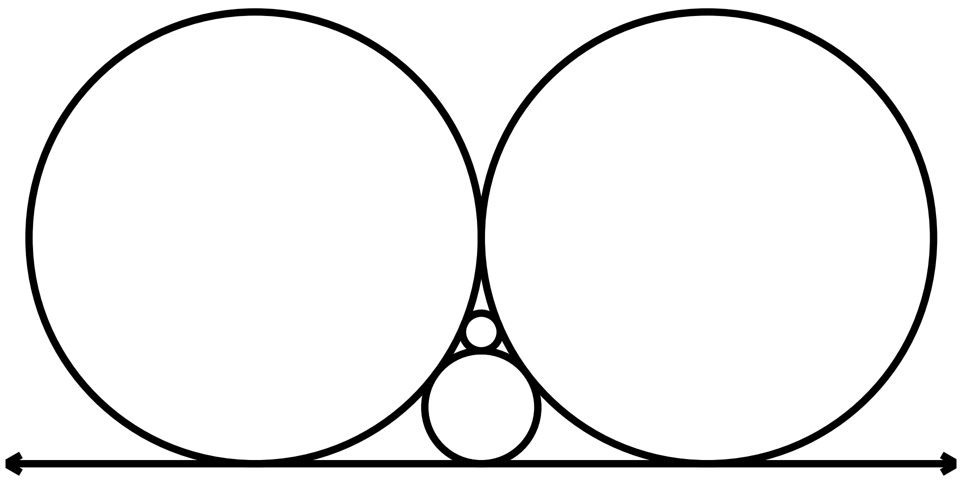
(A) 10395 **(B)** 40320 **(C)** 60125 **(D)** 101245 **(E)** 135135



Solution: (E) Pick one point. There are 13 choices with whom to match it. For any of the remaining 12 points there are 11 choices with whom to match it. Continue in this manner, so we see the total number of ways is $13 \cdot 11\dot{9} \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 135135$.

19. Two congruent large circles and a smaller third circle are mutually externally tangent and are also tangent to the same line, as shown. A fourth circle of diameter one, smaller than the rest, is drawn tangent to these three circles. What is the radius of the two large congruent circles?

(A) $4\sqrt{2}$ **(B)** $31/5$ **(C)** 5 **(D)** $19/3$ **(E)** 6



Solution: (E) Let r be the radius of the two larger circles and s be the radius of the middle size circle. By joining centers and forming right triangles, we obtain the system $(r+1/2)^2 = (r-2s-1/2)^2 + r^2$ and $(r+s)^2 = (r-s)^2 + r^2$. From the second equation, $4rs = r^2$ so $r = 4s$. Substitute this into the first equation yielding after simplification $4s^2 - 6s = 0$ or $s = 3/2$. Thus $r = 4s = 4(3/2) = 6$.

20. Let $\angle ABC$ measure 30 degrees. Imagine the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are silvered as a mirror to reflect light. For a light beam that starts anywhere in the interior of $\angle ABC$, what is the maximum number of times such a beam can strike these mirrors?

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: (D) Join six such wedges together to form a semicircle. A light ray starting in the first wedge continues as a straight line through the semicircle. When it exits the semicircle, no further reflections occur. Thus the maximum is 6 reflections.

Spring 2012 McNabb GDCTM Contest
Algebra II Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. The value of $\left(\sqrt{12}+\sqrt{3}\right)^2$ is

   (A) 36 **(B)** 27 **(C)** 21 **(D)** 15 **(E)** 12

   Solution: (B) Simplify to get $(2\sqrt{3}+\sqrt{3})^2 = (3\sqrt{3})^2 = 9 \cdot 3 = 27$.

2. A cage contains birds and rabbits. There are seventeen heads and forty feet. How many rabbits are in the cage?

   (A) 2 **(B)** 3 **(C)** 4 **(D)** 5 **(E)** 6

   Solution: (B) All birds would mean 34 feet. The extra 6 feet come from 3 rabbits!

3. Mr. and Mrs. Reynolds have three daughters and three sons. At Easter each member of the family buys one chocolate Easter egg for everyone else in the family. How many Easter eggs will the Reynolds family buy in total?

   (A) 28 **(B)** 32 **(C)** 40 **(D)** 56 **(E)** 64

   Solution: (D) Each family member buys 7 eggs, so eight times seven, or 56, eggs were bought in total.

4. If $a \diamond b$ equals the lesser of $1/a$ and $1/b$, find the value of $-3 \diamond (-2 \diamond (-1/2))$.

   (A) -3 **(B)** -1/3 **(C)** -1/2 **(D)** -2 **(E)** -1

   Solution: (C) $-3 \diamond (-2 \diamond (-1/2)) = -3 \diamond -2 = -1/2$.

5. What is the y-intercept of the perpendicular bisector of the segment with endpoints $(-2, 8)$ and $(8, 4)$?

   (A) -1/2 **(B)** 1/2 **(C)** 0 **(D)** -3/2 **(E)** 3/2

   Solution: (D) The slope of the given segment is $(8-4)/(-2-10) = 4/-10 = -2/5$, so the slope perpendicular is $5/2$. The midpoint has coordinates the averages of the endpoints, so $(3, 6)$. The point-slope form of the equation of the perpendicular bisector is then $y - 6 = (5/2)(x-3)$. Substitute $x = 0$ to the find the y-intercept at $-15/2 + 6 = -3/2$.

6. The graph of $z^2 = 4x^2 + 4y^2$ is a double cone and the graph of $2x - z = 8$ is a plane. The intersection of these two graphs is

   (A) a circle **(B)** a non-circular ellipse **(C)** an hyperbola
   (D) a parabola **(E)** two intersecting lines

   Solution: (D) The line satisfying $y = 0$ and $2x = z$ lies on the cone and is parallel to the plane $2x - z = 8$. Thus the conic section is a parabola.

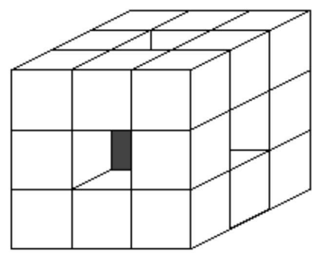
7. Two drovers A and B went to market with cattle. A sold 50 and then had left as many as B, who had not sold any yet. Then B sold 54 and had remaining half as many as A. How many cattle total did they have between them on their way to market?

(A) 104 **(B)** 108 **(C)** 148 **(D)** 158 **(E)** 266

Solution: (E) Two equations result: $A - 50 = B$ and then $B - 54 = (1/2)(A - 50)$. So $B - 54 = (1/2)B$ or $(1/2)B = 54$ or $B = 108$. Thus $A = 158$ and $A + B = 158 + 108 = 266$.

8. Twenty seven small $1 \times 1 \times 1$ cubes are glued together to form a $3 \times 3 \times 3$ cube. Then the center small cube and the small cubes at the center of each face are removed. What is the surface area of the resulting solid?

(A) 56 **(B)** 64 **(C)** 72 **(D)** 84 **(E)** 96



Solution: (C) One way to count is to note that there are 8 corner cubes, showing 3 faces apiece and there are are 12 middle pieces showing 4 faces apiece. Thus the surface of the solid measures $8 \cdot 3 + 12 \cdot 4 = 24 + 48 = 72$.

9. If $2 + \ln x = \ln(x + 2)$ then x must equal

(A) $\dfrac{2}{e^2 - 1}$ **(B)** $\dfrac{2}{e - 1}$ **(C)** $\dfrac{1}{e^2 - 1}$ **(D)** $\dfrac{2}{e^2 + 1}$ **(E)** $\dfrac{1}{e^2 + 1}$

Solution: (A) $2 + \ln x = \ln(e^2) + \ln x = \ln(e^2 x) = \ln(x + 2)$. Since lnx is one-to-one, we have that $e^2 x = x + 2$ so $x = 2/(e^2 - 1)$.

10. Hezy and Zeke are both painters. Working alone Hezy can paint a certain room in 4 hours. Working alone Zeke can paint the same room in 6 hours. Let r_H, r_Z, and r_{HZ} be respectively the rates at which Hezy works alone, Zeke works alone, and Hezy and Zeke work together. Suppose for some coefficient of efficiency k satisfying $0 \leq k \leq 1$, these rates are related by the formula

$$r_{HZ} = k(r_H + r_Z)$$

Very often $k < 1$ since the two working at the same time interfere with each other to some extent. What is the value of k if in fact it takes Hezy and Zeke working together 2.7 hours to paint this room?

(A) 6/7 **(B)** 7/8 **(C)** 8/9 **(D)** 9/10 **(E)** 10/11

Solution: (C) Use rate times time equals distance, to get $k(1/4 + 1/6) \cdot (2.7) = 1$ so $k = 12/(5 \cdot 2.7) = 120/(5 \cdot 27) = 24/27 = 8/9$.

11. If $a^2 - 2a + b^2 - 2b = ab - 4$, then what is the value of $a + 2b$?

(A) 0 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** cannot be uniquely determined

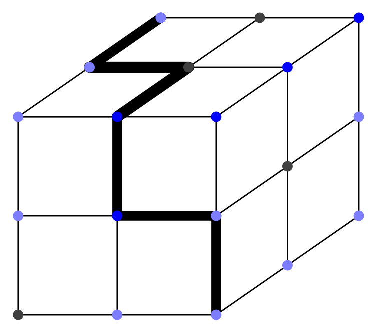
Solution: (B) Double both sides, bring all terms to the left, and complete the squares as follows

$$(a-2)^2 + (b-2)^2 + (a-b)^2 = 0$$

Thus $a = b = 2$ so that $a + 2b = 6$. Note a and b are real as part of the basic assumption of the contest.

12. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

(A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120



Solution: (A) First count all paths, including through the center of the cube. These paths can be labeled with 2 U's, 2 L's, and 2 B's, for up, left, and back moves. The number of 6 letter words of this type is $6!/(2!2!2!) = 720/8 = 90$. Second we count the number of paths through the center. There are 6 ways to get to the center, then 6 ways from the center to the opposite corner, so $6^2 = 36$ paths through the center. Finally, our answer is $90 - 36 = 54$.

13. Find the value of $r^2s^2 + s^2t^2 + t^2r^2$ if r, s, and t are the three possibly complex roots of the cubic polynomial $x^3 + 5x^2 - 3x + 1$.

(A) -1 **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 8

Solution: (A) Note that $r^2s^2 + s^2t^2 + t^2r^2 = (rs + st + tr)^2 - 2(rst)(s + r + t) = 3^2 - 2(-1)(-5) = 9 - 10 = -1$.

14. In $\triangle ABC$ points D, E, and F lie on segments $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$ respectively, in such a way that the proportions $BD/DC = 7/3$, $CE/EA = 3/2$, and $AF/FB = 4/1$ hold. If AD and FE intersect at G, what is the ratio AG/GD?

(A) 5/6 **(B)** 6/7 **(C)** 7/8 **(D)** 8/9 **(E)** 1/1

Solution: (D) Put a mass of 3 at B, a mass of 7 at C, and split masses of 3/4 and 21/2 at A to handle the transversal EF. The center of mass is then G, and will still be G when the mass of 10 is concentrated at D and the mass of $3/4 + 21/2 = 45/4$ at A. Then by the lever principle, $AG \cdot 45/4 = GD \cdot 10$, so that $AG/GD = 10 \cdot 4/45 = 40/45 = 8/9$.

15. If $x > \frac{1}{x}$, then which of the following must be true?

I. $2x > \frac{2}{x}$

II. $2x > \frac{1}{2x}$

III. $x^2 > \frac{1}{x^2}$

(A) I only **(B)** I and II only **(C)** I and III only **(D)** II and III only **(E)** I, II, and III

Solution: (A) Statement I is true, just multiply both sides of the given inequality by 2. Statements II and III are not true in general as may be seen by putting $x = -1/2$ in each.

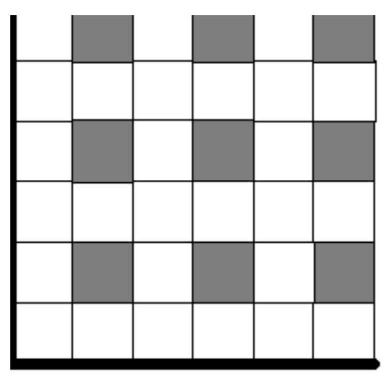
16. When the polynomial x^{2012} is divided by the polynomial $x^2 + x + 1$ what is the remainder $R(x)$?

(A) -1 **(B)** $x + 1$ **(C)** $2x - 1$ **(D)** 0 **(E)** $-x - 1$

Solution: (E) By the division algorithm, $x^{2012} = (x^2 + x + 1)Q(x) + R(x)$ where $R(x) = ax + b$. Though stated for real numbers this identity remains true for complex numbers. So let $x = -1/2 + i\sqrt{3}/2$, a root of $x^2 + x + 1$, and a cube root of 1. For this x, $x^{2012} = x^2 = -1/2 - i\sqrt{3}/2 = a(-1/2 + i\sqrt{3}/2) + b$. Equating real and imaginary parts, $a = -1$ immediately and then $b = -1$ follows. Thus $R(x) = -x - 1$.

17. A rectangular lobby is to be tiled in this pattern in such a way that the border tiles along all the edges of the lobby are white. If the lobby measures 201 tiles by 101 tiles, how many shaded tiles are required?

(A) 4900 **(B)** 4950 **(C)** 5000 **(D)** 5050 **(E)** 5100



Solution: (C) The shaded tiles will be 50 rows of 100 tiles each.

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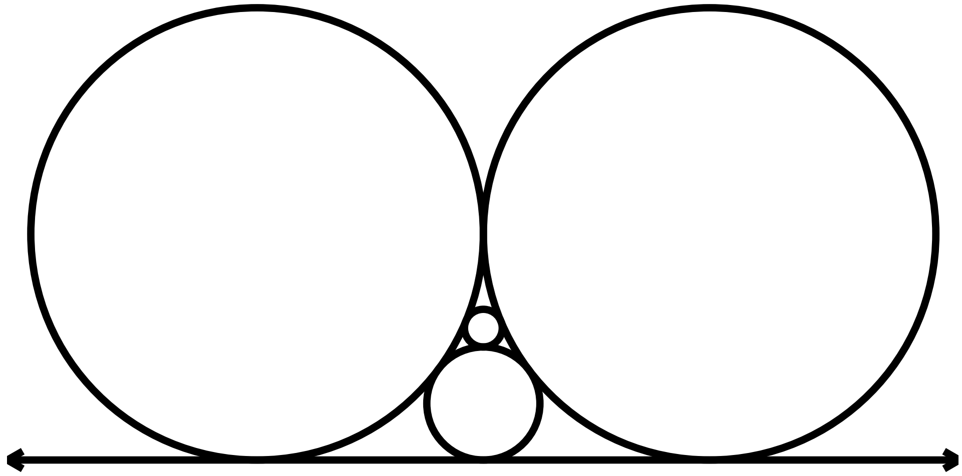
(A) $4\sqrt{2}$ **(B)** $31/5$ **(C)** 5 **(D)** $19/3$ **(E)** 6

Solution: (E) Let r be the radius of the two larger circles and s be the radius of the middle size circle. By joining centers and forming right triangles, we obtain the system $(r + 1/2)^2 = (r - 2s - 1/2)^2 + r^2$ and $(r + s)^2 = (r - s)^2 + r^2$. From the second equation, $4rs = r^2$ so $r = 4s$. Substitute this into the first equation yielding after simplification $4s^2 - 6s = 0$ or $s = 3/2$. Thus $r = 4s = 4(3/2) = 6$.



19. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

    (A) 5/12 **(B)** 1/11 **(C)** 1/24 **(D)** 1/22 **(E)** 1/48

    Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to 1/22.

20. Let $\angle ABC$ measure 30 degrees. Imagine the rays $\overrightarrow{BA}$ and $\overrightarrow{BC}$ are silvered as a mirror to reflect light. For a light beam that starts anywhere in the interior of $\angle ABC$, what is the maximum number of times such a beam can strike these mirrors?

    (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

    Solution: (C) Join six such wedges together to form a semicircle. A light ray starting in the first wedge continues as a straight line through the semicircle. When it exits the semicircle, no further reflections occur. Thus the maximum is 6 reflections.

Spring 2012 McNabb GDCTM Contest
Pre-calculus Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

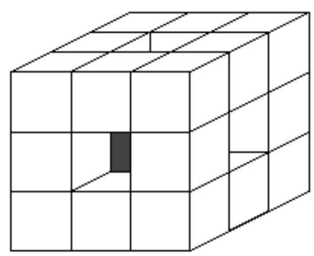
1. The value of $-3-3^2-3^3$ is equal to

   (A) -41 **(B)** -40 **(C)** -39 **(D)** -37 **(E)** -36

   Solution: (C) $-3-9-27=-12-27=-39$.

2. Twenty seven small $1\times 1\times 1$ cubes are glued together to form a $3\times 3\times 3$ cube. Then the center small cube and the small cubes at the center of each face are removed. What is the surface area of the resulting solid?

   (A) 56 **(B)** 64 **(C)** 72 **(D)** 84 **(E)** 96

   Solution: (C) One way to count is to note that there are 8 corner cubes, showing 3 faces apiece and there are are 12 middle pieces showing 4 faces apiece. Thus the surface of the solid measures $8\cdot 3+12\cdot 4=24+48=72$.

3. What is the remainder when the sum

   $$1^{111}+2^{111}+3^{111}+4^{111}+5^{111}+6^{111}+7^{111}+8^{111}+9^{111}+10^{111}$$

   is divided by 11?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

   Solution: (A) Since 10 has the same remainder as -1 when divided by 11, then 10^{111} has the same remainder as $(-1)^{111}$ when divided by 11. So $1^{111}+10^{111}$ has the same remainder as $1^{111}+(-1)^{111}$ or 0 when divided by 11, that is 0. The same works on the other opposite pairs, so the overall remainder is 0 as well.

4. In triangle ABC, put $AB=c$, $BC=a$, and $CA=b$. If $(a+b+c)(a+b-c)=ab$, what is the degree measure of $\angle C$?

   (A) 120 **(B)** 105 **(C)** 90 **(D)** 75 **(E)** 60

   Solution: (A) Rewrite the given constraint as $c^2=a^2+b^2+ab$. Comparison with the Law of Cosines shows that $\cos C=-1/2$ so that $\angle C=120$ degrees.

5. Two drovers A and B went to market with cattle. A sold 50 and then had left as many as B, who had not sold any yet. Then B sold 54 and had remaining half as many as A. How many cattle total did they have between them on their way to market?

(A) 104 **(B)** 108 **(C)** 148 **(D)** 158 **(E)** 266

Solution: (E) Two equations result: $A - 50 = B$ and then $B - 54 = (1/2)(A - 50)$. So $B - 54 = (1/2)B$ or $(1/2)B = 54$ or $B = 108$. Thus $A = 158$ and $A + B = 158 + 108 = 266$.

6. When expanded and simplified $(1 - x + x^2 - x^3)^{10}$ has the form

$$c_0 + c_1 x + c_2 x^2 + \cdots + c_{30} x^{30}$$

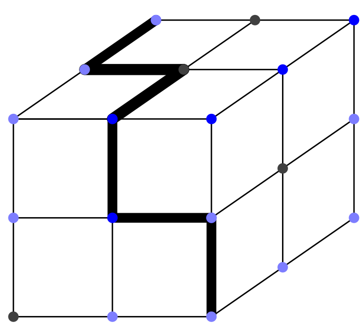
What is the value of $c_1 + c_3 + c_5 + \cdots + c_{29}$, the sum of the coefficients of all the odd powers of x?

(A) -4^{19} **(B)** -2^{20} **(C)** -2^{19} **(D)** 0 **(E)** 2^{19}

Solution: (C) Note that $(1 - x + x^2 - x^3)^{10} = ((1 - x)(1 + x^2))^{10} = (1 - x)^{10}(1 + x^2)^{10}$. Since only even powers of x occur in $(1 + x^2)^{10}$ the only odd powers of x in the final polynomial product must come from the odd powers of x in $(1 - x)^{10}$. Now the sum of all the coefficients in $(1 + x^2)^{10}$ is 2^{10} while the sum of the coefficients of the odd powers of x in $(1 - x)^{10}$ is the opposite of the sum of every other entry in the 10th row of Pascal's triangle, which is known to be half the row sum, so -2^9. Multiplying these gives our answer of -2^{19}.

7. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

(A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120



Solution: (A) First count all paths, including through the center of the cube. These paths can be labeled with 2 U's, 2 L's, and 2 B's, for up, left, and back moves. The number of 6 letter words of this type is $6!/(2!2!2!) = 720/8 = 90$. Second we count the number of paths through the center. There are 6 ways to get to the center, then 6 ways from the center to the opposite corner, so $6^2 = 36$ paths through the center. Finally, our answer is $90 - 36 = 54$.

8. If $\sin x + \cos x = \sqrt{5}/3$, then what is the value of $\sqrt{\cos 4x}$?

(A) $5/9$ **(B)** $7/9$ **(C)** $7/11$ **(D)** $8/11$ **(E)** $12/13$

Solution: (B) Square both sides of the given inequality and use the Pythagorean Identity and the Sine Double Angle Identity to get $\sin 2x = -4/9$. Then $\cos 4x = 1 - 2\sin^2(2x) = 1 - 2(16/81) = 49/81$ so that $\sqrt{\cos 4x} = 7/9$.

9. Find the value of $r^2s^2 + s^2t^2 + t^2r^2$ if r, s, and t are the three possibly complex roots of the cubic polynomial $x^3 + 5x^2 - 3x + 1$.

   (A) -1 **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 8

   Solution: (A) Note that $r^2s^2 + s^2t^2 + t^2r^2 = (rs + st + tr)^2 - 2(rst)(s + r + t) = 3^2 - 2(-1)(-5) = 9 - 10 = -1$.

10. What is the sum of all the odd 5 digit numbers in which each of the digits 1,2,3,4, and 5 occur exactly once?

    (A) 2000976 **(B)** 2188876 **(C)** 2299936 **(D)** 2399976 **(E)** 2499936

    Solution: (D) The sums of all such integers when the last digit is a 1, 3, or 5 can be written respectively as

$$(2+3+4+5)\cdot 6 \cdot 11110 + 24 \cdot 1$$
$$(1+2+4+5)\cdot 6 \cdot 11110 + 24 \cdot 3$$
$$(1+2+3+4)\cdot 6 \cdot 11110 + 24 \cdot 5$$

    Using the distributive property the sum of these three quantities quickly converts into the single product $(66666)(36) = 2399976$.

11. If $a^2 - 2a + b^2 - 2b = ab - 4$, then what is the value of $a + 2b$?

    (A) 0 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** cannot be uniquely determined

    Solution: (B) Double both sides, bring all terms to the left, and complete the squares as follows

$$(a-2)^2 + (b-2)^2 + (a-b)^2 = 0$$

    Thus $a = b = 2$ so that $a + 2b = 6$. Note a and b are real as part of the basic assumption of the contest.

12. Consider the lines $y = 0$, $y = \sqrt{3}$, and $y = x\sqrt{3}$. Let C be the center of the circle tangent to all three lines such that the x-coordinate of this center is negative. The sum of the coordinates of C can be written in the form $a + b\sqrt{3}$ where a and b are rational numbers. Find $a + b$.

    (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

    Solution: (B) The center of the circle lies on the angle bisector of the angle formed by the lines $y = \sqrt{3}$ and $y = \sqrt{3}x$ and on the angle bisector of the angle formed by the lines $y = 0$ and $y = \sqrt{3}x$. Using 30-60-90 triangles the equation of the former is $y - \sqrt{3} = (1/\sqrt{3})(x - 1)$ and the latter $y = -\sqrt{3}x$. These two lines intersect at the center $(-1/2, \sqrt{3}/2)$. Thus $a + b = -1/2 + 1/2 = 0$.

13. The graph of $z^2 = 4x^2 + 4y^2$ is a double cone and the graph of $2x - z = 8$ is a plane. The intersection of these two graphs is

    (A) a circle **(B)** a non-circular ellipse **(C)** an hyperbola
    (D) a parabola **(E)** two intersecting lines

    Solution: (D) The line satisfying $y = 0$ and $2x = z$ lies on the cone and is parallel to the plane $2x - z = 8$. Thus the conic section is a parabola.

14. What is the sum

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{5}+\frac{1}{8}+\frac{1}{10}+\frac{1}{16}+\frac{1}{20}+\frac{1}{25}+\cdots$$

which consists of the sum of the reciprocals of 1 and all the natural numbers whose prime factorizations contain no primes other than 2 or 5?

(A) $5/2$ **(B)** $11/4$ **(C)** $19/8$ **(D)** $17/8$ **(E)** 3

Solution: (A) This series is the product of two positive geometric series as $(1+1/2+1/4+1/8+\cdots)*(1+1/5+1/5^2+1/5^3+\cdots) = 2/(4/5) = 10/4 = 52$. Note that convergent positive term series may be rearranged at will without affecting the sums.

15. If $2+\ln x = \ln(x+2)$ then x must equal

(A) $\frac{2}{e^2-1}$ **(B)** $\frac{2}{e-1}$ **(C)** $\frac{1}{e^2-1}$ **(D)** $\frac{2}{e^2+1}$ **(E)** $\frac{1}{e^2+1}$

Solution: (A) $2+\ln x = \ln(e^2)+\ln x = \ln(e^2 x) = \ln(x+2)$. Since lnx is one-to-one, we have that $e^2 x = x+2$ so $x = 2/(e^2-1)$.

16. In $\triangle ABC$, the medians AD and BE are perpendicular. If $AC = 8$ and $BC = 12$, what is the length of AB?

(A) 6 **(B)** $4\sqrt{13/5}$ **(C)** $4\sqrt{3}$ **(D)** 7 **(E)** 9

Solution: (B) Let the medians meet at centroid G. The medians split each other in a two to one ratio, so if $GD = x$ it follows that $GA = 2x$ and if $GE = y$ then $GB = 2y$. From the Pythagorean theorem, we have the system $x^2+4y^2 = 36$ and $4x^2+y^2 = 16$. Adding we have $5x^2+5y^2 = 52$. But $AB^2 = 4x^2+4y^2 = (4/5)(52) = 16(13/5)$, so $AB = 4\sqrt{13/5}$.

17. When the polynomial x^{2012} is divided by the polynomial x^2+x+1 what is the remainder $R(x)$?

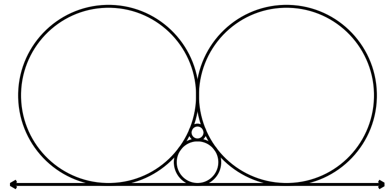
(A) 1 **(B)** $-x-1$ **(C)** $x+1$ **(D)** $2x-1$ **(E)** 0

Solution: (B) By the division algorithm, $x^{2012} = (x^2+x+1)Q(x)+R(x)$ where $R(x) = ax+b$. Though stated for real numbers this identity remains true for complex numbers. So let $x = -1/2+i\sqrt{3}/2$, a root of x^2+x+1, and a cube root of 1. For this x, $x^{2012} = x^2 = -1/2-i\sqrt{3}/2 = a(-1/2+i\sqrt{3}/2)+b$. Equating real and imaginary parts, $a = -1$ immediately and then $b = -1$ follows. Thus $R(x) = -x-1$.

18. A 2012×2012 matrix A has its entry in the ith row and jth column designated by $a_{i,j}$. Suppose for each $k = 1,2,3,\cdots,1006$ that $a_{2k-1,2k-1} = 4k-3$, $a_{2k-1,2k} = 4k-2$, $a_{2k,2k-1} = 4k-1$, and $a_{2k,2k} = 4k$. All other values of $a_{i,j}$ are set equal to zero. Find the value of det A, where det A stands for the determinant of A.

(A) -4^{1006} **(B)** -2^{1006} **(C)** 0 **(D)** 2^{1006} **(E)** 4^{1006}

Solution: (D) Each of the two-by-two block matrices along the main diagonal has determinant equal to -2. The determinant of this entire matrix is the product of the determinant of its two-by-two blocks, so our answer is $(-2)^{1006} = 2^{1006}$.



19. Two congruent large circles and a smaller third circle are mutually externally tangent and also tangent to the same line, as shown. A fourth circle of diameter one, smaller than the rest, is drawn tangent to these three circles. What is the radius of the two large congruent circles?

    (A) 5 **(B)** $4\sqrt{2}$ **(C)** 6 **(D)** 31/5 **(E)** 19/3

    Solution: (C) Let r be the radius of the two larger circles and s be the radius of the middle size circle. By joining centers and forming right triangles, we obtain the system $(r+1/2)^2 = (r-2s-1/2)^2 + r^2$ and $(r+s)^2 = (r-s)^2 + r^2$. From the second equation, $4rs = r^2$ so $r = 4s$. Substitute this into the first equation yielding after simplification $4s^2 - 6s = 0$ or $s = 3/2$. Thus $r = 4s = 4(3/2) = 6$.

20. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

    (A) 5/12 **(B)** 1/11 **(C)** 1/24 **(D)** 1/22 **(E)** 1/48

    Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to 1/22.

Spring 2012 McNabb GDCTM Contest
Calculus Solutions

NO Calculators Allowed

Note: all variables represent real numbers unless otherwise stated in the problem itself.

1. The value of $-3 - 3^2 - 3^3$ is equal to

   (A) -41 **(B)** -40 **(C)** -39 **(D)** -37 **(E)** -36

   Solution: (C) $-3 - 9 - 27 = -12 - 27 = -39.$

2. What is the largest possible product of two positive odd integers whose sum is 40?

   (A) 39 **(B)** 279 **(C)** 300 **(D)** 399 **(E)** 400

   Solution: (D) Make the factors as close together as possible, so choose 19 and 21. Note $19 \cdot 21 = 399$.

3. If $a \diamond b$ equals the lesser of $1/a$ and $1/b$, find the value of $-3 \diamond (-2 \diamond (-1/2))$.

   (A) -3 **(B)** -1/3 **(C)** -1/2 **(D)** -2 **(E)** -1

   Solution: (C) $-3 \diamond (-2 \diamond (-1/2)) = -3 \diamond -2 = -1/2.$

4. What is the remainder when the sum

   $$1^{111} + 2^{111} + 3^{111} + 4^{111} + 5^{111} + 6^{111} + 7^{111} + 8^{111} + 9^{111} + 10^{111}$$

   is divided by 11?

   (A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

   Solution: (A) Since 10 has the same remainder as -1 when divided by 11, then 10^{111} has the same remainder as $(-1)^{111}$ when divided by 11. So $1^{111} + 10^{111}$ has the same remainder as $1^{111} + (-1)^{111}$ or 0 when divided by 11, that is 0. The same works on the other opposite pairs, so the overall remainder is 0 as well.

5. Distribute 14 points along a line segment. How many distinct ways exist for pairing the points via semicircles? The case of four points is pictured below.

   (A) 10395 **(B)** 40320 **(C)** 60125 **(D)** 101245 **(E)** 135135

   Solution: (E) Pick one point. There are 13 choices with whom to match it. For any of the remaining 12 points there are 11 choices with whom to match it. Continue in this manner, so we see the total number of ways is $13 \cdot 11\dot{9} \cdot 7 \cdot 5 \cdot 3 \cdot 1 = 135135$.

6. If $2 + \ln x = \ln(x + 2)$ then x must equal

   (A) $\dfrac{2}{e^2 - 1}$ **(B)** $\dfrac{2}{e - 1}$ **(C)** $\dfrac{1}{e^2 - 1}$ **(D)** $\dfrac{2}{e^2 + 1}$ **(E)** $\dfrac{1}{e^2 + 1}$

   Solution: (A) $2 + \ln x = \ln(e^2) + \ln x = \ln(e^2 x) = \ln(x + 2)$. Since lnx is one-to-one, we have that $e^2 x = x + 2$ so $x = 2/(e^2 - 1)$.

7. When expanded and simplified $(1-x+x^2-x^3)^{10}$ has the form

$$c_0 + c_1x + c_2x^2 + \cdots + c_{30}x^{30}$$

What is the value of $c_1 + c_3 + c_5 + \cdots + c_{29}$, the sum of the coefficients of all the odd powers of x?

(A) -4^{19} **(B)** -2^{20} **(C)** -2^{19} **(D)** 0 **(E)** 2^{19}

Solution: (C) Note that $(1-x+x^2-x^3)^{10} = ((1-x)(1+x^2))^{10} = (1-x)^{10}(1+x^2)^{10}$. Since only even powers of x occur in $(1+x^2)^{10}$ the only odd powers of x in the final polynomial product must come from the odd powers of x in $(1-x)^{10}$. Now the sum of all the coefficients in $(1+x^2)^{10}$ is 2^{10} while the sum of the coefficients of the odd powers of x in $(1-x)^{10}$ is the opposite of the sum of every other entry in the 10th row of Pascal's triangle, which is known to be half the row sum, so -2^9. Multiplying these gives our answer of -2^{19}.

8. A small bug crawls on the surface of a $2 \times 2 \times 2$ cube from one corner to the far opposite corner along the gridlines formed by viewing this cube as an assembly of eight $1 \times 1 \times 1$ cubes. How many shortest paths of this type are possible? One example is shown.

(A) 54 **(B)** 64 **(C)** 90 **(D)** 96 **(E)** 120

Solution: (A) First count all paths, including through the center of the cube. These paths can be labeled with 2 U's, 2 L's, and 2 B's, for up, left, and back moves. The number of 6 letter words of this type is $6!/(2!2!2!) = 720/8 = 90$. Second we count the number of paths through the center. There are 6 ways to get to the center, then 6 ways from the center to the opposite corner, so $6^2 = 36$ paths through the center. Finally, our answer is $90 - 36 = 54$.

9. Find the value of $r^2s^2 + s^2t^2 + t^2r^2$ if r, s, and t are the three possibly complex roots of the cubic polynomial $x^3 + 5x^2 - 3x + 1$.

(A) -1 **(B)** 0 **(C)** 3 **(D)** 5 **(E)** 8

Solution: (A) Note that $r^2s^2 + s^2t^2 + t^2r^2 = (rs+st+tr)^2 - 2(rst)(s+r+t) = 3^2 - 2(-1)(-5) = 9 - 10 = -1$.

10. What is the sum

$$1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8} + \frac{1}{10} + \frac{1}{16} + \frac{1}{20} + \frac{1}{25} + \cdots$$

which consists of the sum of the reciprocals of 1 and all the natural numbers whose prime factorizations contain no primes other than 2 or 5?

(A) 5/2 **(B)** 7/2 **(C)** 5/3 **(D)** 2 **(E)** 3

Solution: (A) This series is the product of two positive geometric series as $(1 + 1/2 + 1/4 + 1/8 + \cdots) * (1 + 1/5 + 1/5^2 + 1/5^3 + \cdots) = 2/(4/5) = 10/4 = 52$. Note that convergent positive term series may be rearranged at will without affecting the sums.

11. In $\triangle ABC$, the medians AD and BE are perpendicular. If $AC = 8$ and $BC = 12$, what is the length of AB?

(A) 6 **(B)** 9 **(C)** $4\sqrt{3}$ **(D)** 7 **(E)** $4\sqrt{13/5}$

Solution: (E) Let the medians meet at centroid G. The medians split each other in a two to one ratio, so if $GD = x$ it follows that $GA = 2x$ and if $GE = y$ then $GB = 2y$. From the Pythagorean theorem, we have the system $x^2 + 4y^2 = 36$ and $4x^2 + y^2 = 16$. Adding we have $5x^2 + 5y^2 = 52$. But $AB^2 = 4x^2 + 4y^2 = (4/5)(52) = 16(13/5)$, so $AB = 4\sqrt{13/5}$.

12. When the polynomial x^{2012} is divided by the polynomial x^2+x+1 what is the remainder $R(x)$?

(A) 1 **(B)** $x+1$ **(C)** $2x-1$ **(D)** 0 **(E)** $-x-1$

Solution: (E) By the division algorithm, $x^{2012} = (x^2+x+1)Q(x)+R(x)$ where $R(x) = ax+b$. Though stated for real numbers this identity remains true for complex numbers. So let $x = -1/2+i\sqrt{3}/2$, a root of x^2+x+1, and a cube root of 1. For this x, $x^{2012} = x^2 = -1/2-i\sqrt{3}/2 = a(-1/2+i\sqrt{3}/2)+b$. Equating real and imaginary parts, $a = -1$ immediately and then $b = -1$ follows. Thus $R(x) = -x-1$.

13. Which of the following are equal to $\int_a^b f(x)\,dx$ for all continuous functions f and all values of the constants a, b, and k, with $k \neq 0$?

I. $-\int_b^a f(x)\,dx$

II. $\int_a^b f(a+b-x)\,dx$

III. $\int_{ka}^{kb} f(kx)\,dx$.

(A) none of them **(B)** I only **(C)** I and II only
(D) I and III only **(E)** I, II and III

Solution: (C) Statement I is a well known property of integrals. Statement II follows from the substitution $u = a+b-x$. Statement III is false because if in it we make $u = kx$ then $du = kdx$ and this extra factor of k is missing.

14. When the integrals listed below are arranged in order from least to greatest, which integral will be in the center?

$$\int_0^1 1+x\,dx \qquad \int_0^1 \frac{1}{1+x}\,dx \qquad \int_0^1 \frac{1}{1+x+\frac{x^2}{2}}\,dx \qquad \int_0^1 e^x\,dx \qquad \int_0^1 e^{-x}\,dx$$

(A) $\int_0^1 1+x\,dx$ **(B)** $\int_0^1 \frac{1}{1+x}\,dx$ **(C)** $\int_0^1 \frac{1}{1+x+\frac{x^2}{2}}\,dx$
(D) $\int_0^1 e^x\,dx$ **(E)** $\int_0^1 e^{-x}\,dx$

Solution: (B) For $x > 0$, using the power series for e^x and basic inequalities it follows that

$$e^{-x} < \frac{1}{1+x+\frac{x^2}{2}} < \frac{1}{1+x} < 1+x < e^x$$

Thus the middle integral will be $\int_0^1 \frac{1}{1+x}\,dx$.

15. If $f(x)$ and $g(x)$ are differentiable functions and $F(x) = \int_0^{g(x)} f(t)\,dt$, then $F'(1)$ equals

(A) $f(g(1))$ **(B)** $f(1)g'(1)$ **(C)** $f'(g(1))g'(1)$ **(D)** $f(g(1))g'(1)$ **(E)** $f'(1)g'(1)$

Solution: (D) From the FTC and the chain rule.

16. Evaluate $\displaystyle\int_0^{49} \frac{1}{\sqrt{16-\sqrt{x}}}\,dx.$

(A) 28/3 **(B)** 32/3 **(C)** 12 **(D)** 40/3 **(E)** 44/3

Solution: (E) Substitute $u = 16-\sqrt{x}$, so $du = -1/(2\sqrt{x})\,dx$ or $dx = -2(16-u)$ to transform the given integral to $2\int_9^{16} \frac{16-u}{\sqrt{u}}\,du = 2\int_9^{16} 16u^{-1/2} - u^{1/2}\,du$ which can now be easily done to give our answer of 44/3.

17. Given that the area of an ellipse with semimajor axis a and semiminor axis b is πab, find the volume of the set of points $\{(x,y,z) : \frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{25} \leq 1\}$.

(A) 10π **(B)** 20π **(C)** 30π **(D)** 40π **(E)** 50π

Solution: (D) Slice perpendicular to the z axis at height z to create an elliptical cross-section with $a = 2\sqrt{1 - z^2/25}$ and $b = 3\sqrt{1 - z^2/25}$, so that $V = 2\pi \int_0^5 2 \cdot 3 \cdot (1 - z^2/25)\, dz = 40\pi$.

18. John has just learned the arclength formula for functions of the form $y = f(x)$ and wishes to test this formula by measuring with a string the actual graph of $y = x^2$ from $x = 0$ to $x = 1$, with one unit in both the x and y direction measuring one inch. However, when he prints the graph of $y = x^2$ using his Computer Algebra System, he notes that while the unit in the x direction does measure exactly one inch, the unit in the y direction measures only three-quarters of an inch. Which integral below will give the length in inches of the actual printed curve $y = x^2$ from $x = 0$ to $x = 1$?

(A) $\int_0^1 \sqrt{1 + \frac{9}{4}x^2}\, dx$ **(B)** $\int_0^1 \sqrt{1 + \frac{4}{9}x^2}\, dx$ **(C)** $\int_0^1 \sqrt{1 + 4x^2}\, dx$
(D) $\int_0^1 \sqrt{1 + 9x^2}\, dx$ **(E)** $\int_0^1 \sqrt{1 + 6x^2}\, dx$

Solution: (A) The equation of the curve as printed is a parabola with vertex at $(0,0)$ and passing through the point $(1, 3/4)$, which has equation $y = 3x^2/4$. Then $y' = 3x/2$ so the arc length formula gives $L = \int_0^1 \sqrt{1 + \frac{9}{4}x^2}\, dx$.

19. Given that $\ln 2 = 1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + \cdots$ and $\ln k = 1 + 1/2 + 1/3 - 3/4 + 1/5 + 1/6 + 1/7 - 3/8 + \cdots + 1/(4n+1) + 1/(4n+2) + 1/(4n+3) - 3/(4n+4) + \cdots$, then $k =$

(A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

Solution: (B) Note that $\ln k = 1 - 1/2 + 1 + 1/3 - 1/4 - 1/2 + 1/5 - 1/6 + 1/3 + 1/7 - 1/8 - 1/4 + \cdots = (1 - 1/2 + 1/3 - 1/4 + 1/5 - 1/6 + 1/7 + \cdots) + (1 - 1/2 + 1/3 - 1/4 + \cdots) = \ln 2 + \ln 2 = \ln 4$. To justify this rearrangement note that if S_n is the nth partial sum for the $\ln 2$ series and T_n is the nth partial sum of the $\ln k$ series, then $T_{4n} = S_{4n} + S_{2n}$ so that $\lim_{n\to\infty} T_{4n} = \ln 2 + \ln 2 = \ln 4$. To complete the argument, $T_{4n+1} = T_{4n} + 1/(4n+1)$, $T_{4n+2} = T_{4n} + 1/(4n+1) + 1/(4n+2)$, and $T_{4n+3} = T_{4n} + 1/(4n+1) + 1/(4n+2) + 1/(4n+3)$, so that the remaining mod 4 subsequences all have also the limit $\ln 4$, establishing that the limit of T_n is again $\ln 4$.

20. A large circular metal plate has 12 equal smaller circular holes drilled out along its periphery to hold test tubes. Currently the plate holds no test tubes, but soon a robot arm will randomly place 5 test tubes on the plate. What is the probability that after all 5 of these test tubes are placed no two test tubes will be adjacent to one another?

(A) $5/12$ **(B)** $1/11$ **(C)** $1/24$ **(D)** $1/22$ **(E)** $1/48$

Solution: (D) Number the holes 1 through 12 and letter the tubes A, B, C, D, and E. Place the E tube in hole 1. Now there are $11 \cdot 10 \cdot 9 \cdot 8$ ways to place the remaining tubes. Of these we now count the number that will have no two adjacent. First holes 2 and 12 cannot be used. The holes 3 thru 11 make up 9 possible locations of which 4 will be taken, leaving 5 as buffers. These 5 buffers have 6 spaces around them, 4 in between, and two at the ends. From these 6 spaces, 4 must be chosen for the 4 test tubes still to be placed. And then there are 4! ways to place those 4 tubes in those spaces. Thus there are $\binom{6}{4}4! = 15 \cdot 24$ ways to place the remaining tubes so no two tubes are adjacent. Thus the probability of doing this is $\frac{15 \cdot 24}{11 \cdot 10 \cdot 9 \cdot 8}$, which simplifies to $1/22$.