Fall 2010 McNabb GDCTM Contest
Pre-Algebra Solutions

NO Calculators Allowed

1. Zeke's piggy bank has 111 coins. If it contains an equal number of quarters, dimes and nickels, and no other type of coin, the total value of the coins in his piggy bank is

   (A) \$14.80 **(B)** \$20.40 **(C)** \$24.00 **(D)** \$27.00 **(E)** \$44.40

   Solution: (A) Since $111 = 3 \cdot 37$ there are 37 groups of coins each worth \$0.40. Then $37(.4) = 14.80$.

2. Keith's grandfather paid 70% of the cost of a jacket. Keith paid the rest. If Keith paid \$39 how much did his grandfather pay?

   (A) \$70 **(B)** \$91 **(C)** \$113 **(D)** \$130 **(E)** \$140

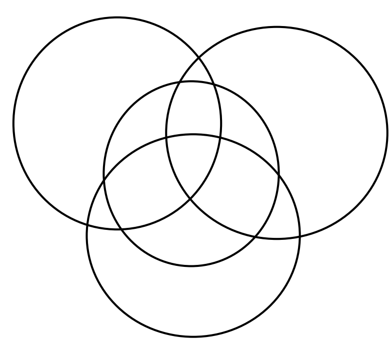
   Solution: (B) Let x be the amount paid by the grandfather. Then $x/39 = 7/3$ and $x = (7/3)39 = 91$.

3. The remainder when 5^{2010} is divided by 7 equals

   (A) 1 **(B)** 2 **(C)** 3 **(D)** 5 **(E)** 6

   Solution: (A) When the powers 5^1, 5^2, 5^3, 5^4, 5^5, and 5^6 are divided by 7 the remainders are 5, 4, 6, 2, 3, and 1 respectively. This cycle of length 6 then repeats. When 2010 is divided by 6 the remainder is zero, indicating that the remainder for 5^{2010} is the same as for 5^6.

4. Four circles are drawn, all in the same plane. Find the maximum number of regions they can form. The diagram shows how four circles may form 8 regions.



   (A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: (C) Two circles intersect at most twice. The maximum number of regions occurs when each of the four circles intersects each of the others twice as shown.

5. If the square root of a positive number falls between seven and eight, then the cube root of this number must fall between

(A) 7 and 8 **(B)** 6 and 7 **(C)** 5 and 6
(D) 4 and 5 **(E)** 3 and 4

Solution: (E) Let a be the positive number. Then $49 < a < 64$, which implies $27 < a < 64$ so $3 < \sqrt[3]{a} < 4$.

6. What is the largest prime number which is a factor of every six digit number of the form $\underline{ABABAB}$?

(A) 3 **(B)** 7 **(C)** 37 **(D)** 41 **(E)** 101

Solution: (C) Since $\underline{ABABAB} = 10101 \cdot \underline{AB}$, we seek the largest prime factor of 10101. From $10101 = 3 \cdot 7 \cdot 13 \cdot 37$, the answer is 37.

7. John has 54 coins totaling one dollar in value. Some are pennies; some are nickels; some are dimes. He has no other kind of coin. How many nickels does John have?

(A) 7 **(B)** 9 **(C)** 11 **(D)** 13 **(E)** 15

Solution: (A) Note that 45 pennies, 2 dimes, and 7 nickels does work. The only number of pennies that have a chance of working are 50, 45, and 40. Replacing a dime by two nickels keeps the value of the coins the same but increases the number of coins by one so with a given number of pennies only one configuration of nickels and dimes can work. Now 50 pennies and 5 dimes is already too many coins. And 40 pennies and 12 nickels is too few coins. So there is only one way to satisfy the constraints of the problem.

8. How many lines of symmetry does a regular octagon have?

(A) 1 **(B)** 2 **(C)** 4 **(D)** 8 **(E)** 16

Solution: (D) There are 8. Four join opposite vertices while four others join the midpoints of opposite edges.

9. Seven consecutive integers are written on a whiteboard. When one of them is erased, the sum of the remaining six integers is 4208. What is the sum of the seven integers?

(A) 4893 **(B)** 4900 **(C)** 4907 **(D)** 4914 **(E)** 4921

Solution: (C) The only integer within $4/7$ of $4208/6$ is 701. This integer must be the mean of the seven. The sum $698 + 699 + 700 + 701 + 702 + 703 + 704$ is 4907. The integer erased was 699.

10. The next three Ranger batters get a hit with probabilities equal to 0.250, 0.300, and 0.500 respectively. What is the probability that all three get hits?

(A) 1.000 **(B)** .300 **(C)** .125 **(D)** .05 **(E)** .0375

Solution: (E) This probability is the product $(0.250)(0.300)(0.500) = .0375$.

11. A positive integer has the interesting property that when expressed as a three digit base-7 number, those digits are the reverse of its digits when expressed as a base-9 number. What is this number expressed in normal form as a base-10 number?

(A) 124 **(B)** 241 **(C)** 248 **(D)** 428 **(E)** 503

Solution: (C) Let $\underline{ABC}$ be the base-7 digits of the integer sought. Then $49A + 7B + C = 81C + 9B + A$ or $B = 24A - 40C$. The digit B must be drawn from the digits 0 through 6. When A is 2 and C is 1, B is too large. The next possible answer occurs when A is 5 and C is 3. This would force B to equal 0 and so works. No other combination works. So the number is base-7 is 503. In base-10 this comes to 248.

12. On a certain island, there are currently 1000 inhabitants, and 91% of these inhabitants were born there. Then some of these native inhabitants leave, so that now only 90% of the inhabitants of the island were born there. Assuming no other kind of change (births, deaths, immigration, etc...) in the population took place, how many of the native inhabitants left?

(A) 9 **(B)** 10 **(C)** 40 **(D)** 90 **(E)** 100

Solution: (E) When 100 natives leave, there are 810 natives left out of 900 inhabitants and $810/900 = 9/10 = 90\%$.

13. How many positive integers less than 200 have an odd number of factors?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 14

Solution: (E) Only perfect squares have an odd number of factors and $14^2 = 196$ is the largest perfect square less than 200.

14. In how many ways can the the letters in *syzygy* be arranged so that the three y's do not all occur together?

(A) 96 **(B)** 112 **(C)** 113 **(D)** 114 **(E)** 120

Solution: (A) Count the complement. When the three y's are together there are four ways to position this block of y's and 3! to scramble the remaining letters. So there are 24 ways to not accomplish the given task. Thus there are $6!/3! - 24 = 120 - 24 = 96$ ways to accomplish it.

15. What is the greatest possible area of a triangle if two of its sides measure 8 and 13?

(A) 41 **(B)** 47 **(C)** 48 **(D)** 51 **(E)** 52

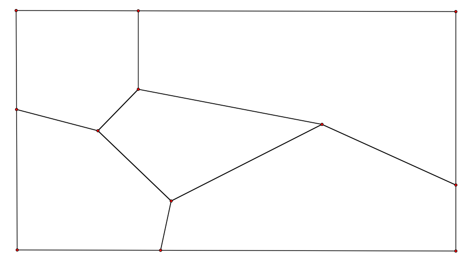
Solution: (E) The maximum area occurs when the sides of 8 and 13 are perpendicular to each other. This triangle has area $(1/2)(8)(13) = 52$.

16. For her Math Club's fundraiser, Carla biked 8 miles south, then 8 miles west, and finally 7 miles south. How many miles, as the crow flies, was she from her start point?

(A) 14 **(B)** 15 **(C)** 17 **(D)** 20 **(E)** 23

Solution: (C) The answer is the length of the hypotenuse of an 8-15-17 right triangle.

17. A map maker has four colors available to color this map consisting of 5 counties. Each county is colored with a single color. No two counties that share a common boundary may be colored the same. In how many ways can our map maker color this map?



(A) 48 **(B)** 60 **(C)** 72 **(D)** 96 **(E)** 120

Solution: (C) There are 4 ways to color the middle county and then 3 to color the upper left county. For the lower left and upper right counties, either they have the same color or they do not. If they do have the same color there are 2 choices for that color and then 2 choices to color the lower right county. If they do not have the same color there are two choices for one of them and the other is forced, leaving the lower right forced as well. So altogether there are $12(4+2) = 72$.

18. Points A, B, C, and D are collinear and occur in the same order as given. If the ratio $AB : BC$ equals 3 and the ratio $BD : AB$ equals $8/3$, then determine the ratio $CD : AC$.

(A) $\frac{5}{3}$ **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{7}{4}$ **(E)** $\frac{8}{3}$

Solution: (D) Let $x = BC$, then $AB = 3x$, and $CD = 7x$. So $CD/AC = 7x/(4x) = 7/4$

19. What is the smallest possible product of three distinct numbers chosen from the set $\{-3, -2, -1, 0, 1, 3\}$

(A) -18 **(B)** -9 **(C)** -6 **(D)** 0 **(E)** 2

Solution: (B) Note $-9 = (-3)(3)(1)$.

20. The height of a square pyramid is increased by 60% and the sides of its base are decreased by 20%. By what percent is the volume of the pyramid increased?

(A) 40% **(B)** 4.8% **(C)** 4% **(D)** 2.4% **(E)** 0%

Solution: (D) Note $1.6(.8)^2 = 1.024$ corresponding is an increase of 2.4%.

21. A baseball team has won 50 games out of 75 so far played. If there are 45 games yet to be played, how many of these must be won in order for the team to finish its season having won exactly 60% of its games?

(A) 18 **(B)** 19 **(C)** 20 **(D)** 21 **(E)** 22

Solution: (E) Note that 60% of 120 is 72. So 22 games are still to be won!

22. How many factors of 720 are also factors of 630?

(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 16

Solution: (C) To be a factor of both 720 and 630 is to be a factor of their greatest common factor 90, and vice versa. And 90 has 12 factors.

23. The value of the fraction

$$\frac{3+6+9+\cdots+99}{4+8+12+\cdots+132}$$

is

(A) $2/3$ **(B)** $3/4$ **(C)** $4/5$ **(D)** $5/6$ **(E)** 7

Solution: (B) Note the ratio of the kth term of the numerator to the kth term of the denominator is always $3/4$. Since both numerator and denominator have the same number of terms, this ratio is maintained.

24. To specify the order of operations in multiplying 5 numbers together, three sets of parentheses are needed. Two ways, for example, are $((ab)(cd))e$ and $(((ab)c)d)e$. In how many ways can these three sets of parentheses be arranged? Assume the order of the numbers a through e is never changed.

(A) 12 **(B)** 13 **(C)** 14 **(D)** 15 **(E)** 16

Solution: (C) There are 14 ways. The number of ways for grouping n numbers in a product are given by the Catalan numbers.

25. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

(A) 32 **(B)** 28 **(C)** 24 **(D)** 20 **(E)** 16

Solution: (A) Note that by symmetry there are an equal number of either foot. There are 16 that start on the increasing foot. They are in order: 13254, 14253, 14352, 15243, 15342, 24153, 24351, 25143, 25341, 34152, 34251, 35142, 35241, 35412, 45132, and 45231. What 1-1 correspondence makes this symmetry explicit?

Fall 2010 McNabb GDCTM Contest
Algebra I Solutions

NO Calculators Allowed

1. The algebraic expression

$$(a - b - c) - (a + b - c) - (a - b + c) - (-a + b - c)$$

is equivalent to

(A) $a + b + c$ **(B)** $-a$ **(C)** $-2a$ **(D)** $-2b$ **(E)** $b - c$

Solution: (D) The a's and c's cancel out.

2. An automobile goes $y/9$ yards in d seconds. How many feet does it travel in two minutes time?

(A) $\frac{40y}{d}$ **(B)** $\frac{40d}{3y}$ **(C)** $\frac{3y}{40d}$ **(D)** $120yd$ **(E)** $\frac{120y}{d}$

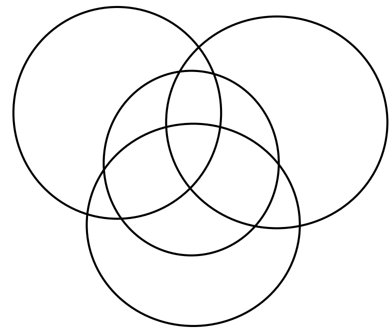
Solution: (A) In one second the auto goes $y/9d$ yards or $y/3d$ feet. In 120 seconds the auto goes $120 \cdot y/3d$ or $40y/d$ feet.

3. The well-known formula $f = (9/5)c + 32$ relates the temperature f in Fahrenheit to the temperature c in Celsius. For how many values of f satisfying $32 \le f \le 212$, will the temperature be an integer in both of these scales?

(A) 9 **(B)** 10 **(C)** 19 **(D)** 20 **(E)** 21

Solution: (E) These conditions are satisfied precisely when $c = 0, 5, 10, 15, \cdots, 100$, or 21 times.

4. Four circles are drawn, all in the same plane. Find the maximum number of regions they can form. The diagram shows how four circles may form 8 regions.



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(A) 7 and 8 **(B)** 6 and 7 **(C)** 5 and 6
(D) 4 and 5 **(E)** 3 and 4

Solution: (E) Let a be the positive number. Then $49 < a < 64$, which implies $27 < a < 64$ so $3 < \sqrt[3]{a} < 4$.

6. John has 54 coins totaling one dollar in value. Some are pennies; some are nickels; some are dimes. He has no other kind of coin. How many nickels does John have?

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Solution: (A) Note that 45 pennies, 2 dimes, and 7 nickels does work. The only number of pennies that have a chance of working are 50, 45, and 40. Replacing a dime by two nickels keeps the value of the coins the same but increases the number of coins by one so with a given number of pennies only one configuration of nickels and dimes can work. Now 50 pennies and 5 dimes is already too many coins. And 40 pennies and 12 nickels is too few coins. So there is only one way to satisfy the constraints of the problem.

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10. At Zeke's Zucchini Stand, 3 zucchini's and 2 squash cost \$4.75, while 2 zucchini's and 3 squash cost \$5.25. How much would 3 zucchini's and 3 squash cost?

(A) \$5.50 **(B)** \$5.75 **(C)** \$6 **(D)** \$6.25 **(E)** \$6.50

Solution: (C) Add the two given purchases together to obtain that 5 zucchini's and 5 squash cost 10 dollars. So three of each must cost 6 dollars.

11. Two sides of a right triangle have lengths 6 and 8 respectively. The product of all the possible lengths of the third side can be written in the form $\sqrt{N}$, for some integer N. What is N?

(A) 100 **(B)** 2400 **(C)** 2800 **(D)** 3200 **(E)** 3600

Solution: (C) Clearly 10 is one possible third side. The only other is $\sqrt{64-36} = \sqrt{28}$.

12. In how many ways can the the letters in *syzygy* be arranged so that the three y's do not all occur together?

(A) 96 **(B)** 112 **(C)** 113 **(D)** 114 **(E)** 120

Solution: (A) Count the complement. When the three y's are together there are four ways to position this block of y's and 3! to scramble the remaining letters. So there are 24 ways to not accomplish the given task. Thus there are $6!/3! - 24 = 120 - 24 = 96$ ways to accomplish it.

13. Points A, B, C, and D are collinear and occur in the same order as given. If the ratio $AB : BC$ equals 3 and the ratio $BD : AB$ equals $8/3$, then determine the ratio $CD : AC$.

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Solution: (D) Let $x = BC$, then $AB = 3x$, and $CD = 7x$. So $CD/AC = 7x/(4x) = 7/4$

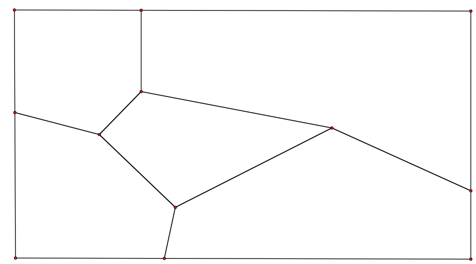
14. A baseball team has won 50 games out of 75 so far played. If there are 45 games yet to be played, how many of these must be won in order for the

team to finish its season having won exactly 60% of its games?

(A) **(B)** **(C)** 22 **(D)** **(E)**

Solution: (C) Note that 60% of 120 is 72. So 22 games are still to be won!

15. A map maker has four colors available to color this map consisting of 5 counties. Each county is colored with a single color. No two counties that share a common boundary may be colored the same. In how many ways can our map maker color this map?



(A) 36 **(B)** 48 **(C)** 72 **(D)** 96 **(E)** 120

Solution: (C) There are 4 ways to color the middle county and then 3 to color the upper left county. For the lower left and upper right counties, either they have the same color or they do not. If they do have the same color there are 2 choices for that color and then 2 choices to color the lower right county. If they do not have the same color there are two choices for one of them and the other is forced, leaving the lower right forced as well. So altogether there are $12(4+2) = 72$.

16. The value of the fraction

$$\frac{3+6+9+\cdots+99}{4+8+12+\cdots+132}$$

is

(A) 2/3 **(B)** 3/4 **(C)** 4/5 **(D)** 5/6 **(E)** 7

Solution: (B) Note the ratio of the kth term of the numerator to the kth term of the denominator is always 3/4. Since both numerator and denominator have the same number of terms, this ratio is maintained.

17. A bag contains 4 quarters and 2 dimes. If 3 coins are randomly removed from the bag, what is the expected total value in cents of these three coins?

(A) 50 **(B)** 55 **(C)** 60 **(D)** 65 **(E)** 75

Solution: (C) The probability of drawing 3 quarters is $1/5$; of drawing two quarters is $3/5$; of one quarter is $1/5$. The expected value of the draw is $75(1/5) + 60(3/5) + 45(1/5) = 60$.

18. The midpoints of the sides of a triangle are $(7, 4)$, $(1, 2)$, and $(1, 6)$. What is the area of this triangle?

(A) 12 **(B)** 24 **(C)** 30 **(D)** 36 **(E)** 48

Solution: (E) The area of the triangle is 4 times the area of the triangle formed by its midpoints. The base of the midpoint triangle is 4 and its height is 6, giving an area of 12. The original triangle then has area 48.

19. The set S contains seven numbers whose mean is 202. The mean of the four smallest numbers in S equals 100, while the mean of the four largest numbers in S equals 300. What is the median of all the numbers in S?

(A) 184 **(B)** 186 **(C)** 192 **(D)** 196 **(E)** 200

Solution: (B) The sum of all the numbers is 1414; of the four smallest is 400; of the four largest is 1200. Adding the last two numbers exceeds 1414 by 186. This excess must be the median of all seven as it was double counted.

20. To specify the order of operations in multiplying 5 numbers together, three sets of parentheses are needed. Two ways, for example, are $((ab)(cd))e$ and $(((ab)c)d)e$. In how many ways can these three sets of parentheses be arranged? Assume the order of the numbers a through e is never changed.

(A) 14 **(B)** 15 **(C)** 16 **(D)** 17 **(E)** 18

Solution: (A) There are 14 ways. The number of ways for grouping n numbers in a product are given by the Catalan numbers.

21. Let $f(x) = |x+1| - |x| + |x-1|$. What is the minimum value of $f(x)$?

(A) -2 **(B)** -1 **(C)** 0 **(D)** 1 **(E)** 2

Solution: (D) For $|x|$ large, $f(x)$ is large too, so a minimum exists. As $f(x)$ is piecewise linear, its minimum must occur at a corner. Corners only occur at $x = -1, 0, 1$. Plugging these into $f(x)$ we see that the minimum value is 1.

22. Let a, b, and c be positive integers with $LCM(a,b) = 48$, $LCM(b,c) = 42$, and $LCM(c,a) = 112$ then the value of $LCM(a,b,c)$ is

(A) 224 **(B)** 336 **(C)** 448 **(D)** 672 **(E)** cannot be determined

Solution: (B) The LCM is found by taking the product of the highest powers of each given prime that occurs among the numbers. Since the highest among those taken two at a time is also the highest period then $LCM(a,b,c) = LCM(48, 42, 112) = 336$.

23. Find the area of quadrilateral $ABCD$ given that $AB = 13$, $BC = 10$, $CD = 10$, $DA = 13$, and $AC = 13$.

(A) 80 **(B)** 90 **(C)** 100 **(D)** 110 **(E)** 120

Solution: (E) Draw segments from A to the midpoints of BC and CD. This splits quadrilateral $ABCD$ into 4 right triangles with sides 5,12, and 13. So the answer is $4(12)(5)(1/2) = 120$.

24. While Xerxes marched on Greece his army streched out for 50 miles. A dispatch rider had to ride from the rear to the head of the army, then instantly turn about and return to the rear. While he did this, the army advanced 50 miles. How many miles did the rider ride?

(A) 100 **(B)** 150 **(C)** $50 + 100\sqrt{2}$ **(D)** $100\sqrt{2}$ **(E)** $50 + 50\sqrt{2}$

Solution: (E) Let x be the distance the army advanced while the rider went to the front. Then $(50+x)/x = x/(50-x)$, since distance is proportional to speed in over equal time intervals. Thus $x = \sqrt{1250} = 25\sqrt{2}$. So the rider travelled $50 + 2x = 50 + 50\sqrt{2}$ miles.

25. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

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Fall 2010 McNabb GDCTM Contest Geometry Solutions

NO Calculators Allowed

1. An automobile goes $y/9$ yards in d seconds. How many feet does it travel in two minutes time?

   (A) $\frac{40y}{d}$ **(B)** $\frac{40d}{3y}$ **(C)** $\frac{3y}{40d}$ **(D)** $120yd$ **(E)** $\frac{120y}{d}$

   Solution: (A) In one second the auto goes $y/9d$ yards or $y/3d$ feet. In 120 seconds the auto goes $120 \cdot y/3d$ or $40y/d$ feet.

2. The well-known formula $f = (9/5)c + 32$ relates the temperature f in Fahrenheit to the temperature c in Celcius. For how many values of f satisfying $32 \leq f \leq 212$, will the temperature be an integer in both of these scales?

   (A) 9 **(B)** 10 **(C)** 19 **(D)** 20 **(E)** 21

   Solution: (E) These conditions are satisfied precisely when $c = 0, 5, 10, 15, \cdots, 100$, or 21 times.

3. Suppose the *converse* of the following statement is true:

   If Zerb is from Xanlor, then Zerb is blue.

   Which of the following statements **must** be true?

   I. If Zerb is not from Xanlor, then Zerb is not blue.
   II. If Zerb is from Xanlor, then Zerb is blue.
   III. If Zerb is blue, then Zerb is not from Xanlor.

   (A) I only **(B)** II only **(C)** III only
   (D) I and II only **(E)** I and III only

   Solution: (A) The converse reads: If Zerb is blue then Zerb is from Xanlor, so that statement I is its contrapositive, and thus true. Statements II and III are not necessarily true.

4. At Zeke's Zucchini Stand, 3 zucchini's and 2 squash cost \$4.75, while 2 zucchini's and 3 squash cost \$5.25. How much would 3 zucchini's and 3 squash cost?

   (A) \$5.50 **(B)** \$5.75 **(C)** \$6 **(D)** \$6.25 **(E)** \$6.50

   Solution: (C) Add the two given purchases together to obtain that 5 zucchini's and 5 squash cost 10 dollars. So three of each must cost 6 dollars.

5. A square is inscribed in a right triangle with sides of length 3, 4, and 5, so that one of the sides of the square is contained in the hypotenuse of the right triangle. What is the side length of the square?

   (A) $\frac{60}{37}$ **(B)** 2 **(C)** $\frac{12}{5}$ **(D)** 3 **(E)** cannot be determined

   Solution: (A) Draw the 3-4-5 triangle with the hypotenuse as base. Then sitting atop the inscribed square is a right triangle with sides $3y$, $4y$, and $5y$, with $5y$ also being the side length of the square. By similar triangles, $(4-4y) : 5y = 5 : 3$. Solving this proportion yields $y = 12/37$ so that $5y = 60/37$.

6. In how many ways can the the letters in the string $ABECEDA$ be arranged so that the consonants are in alphabetical order?

   (A) 90 **(B)** 105 **(C)** 120 **(D)** 180 **(E)** 210

   Solution: (E) Given an initial arrangement of the letters, there are six ways to scramble the 3 consonants (including the original arrangement) only one of which has the consonants in alphabetical order. Thus one-sixty of all possible arrangements of these letters is what we seek. Our answer is then $(1/6)7!/(2!2!) = 210$.

7. Points A, B, C, and D are collinear and occur in the same order as given. If the ratio $AB : BC$ equals 3 and the ratio $BD : AB$ equals $8/3$, then determine the ratio $CD : AC$.

   (A) $\frac{5}{3}$ **(B)** $\frac{3}{2}$ **(C)** 2 **(D)** $\frac{7}{4}$ **(E)** $\frac{8}{3}$

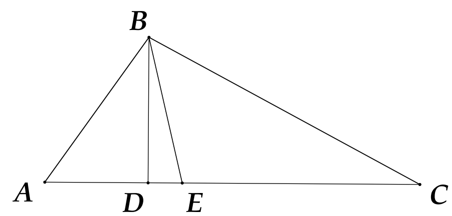
   Solution: (D) Let $x = BC$, then $AB = 3x$, and $CD = 7x$. So $CD/AC = 7x/(4x) = 7/4$

8. A baseball team has won 50 games out of 75 so far played. If there are 45 games yet to be played, how many of these must be won in order for the team to finish its season having won exactly 60% of its games?

(A) 20 **(B)** 21 **(C)** 22 **(D)** 23 **(E)** 72

Solution: (C) Note that 60% of 120 is 72. So 22 games are still to be won!

9. In $\triangle ABC$, $\angle A = 60°$, $\angle C = 40°$, $BD \perp AC$ and $\overrightarrow{BE}$ bisects $\angle ABC$. Find the measure of $\angle DBE$ in degrees.



(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 20

Solution: (B) Note $\angle B = 80$ so $\angle ABE = 40$. But $\angle ABD = 30$, so that $\angle DBE = 40 - 30 = 10$.

10. A bag contains 4 quarters and 2 dimes. If 3 coins are randomly removed from the bag, what is the expected total value in cents of these three coins?

(A) 50 **(B)** 55 **(C)** 60 **(D)** 65 **(E)** 75

Solution: (C) The probability of drawing 3 quarters is $1/5$; of drawing two quarters is $3/5$; of one quarter is $1/5$. The expected value of the draw is $75(1/5) + 60(3/5) + 45(1/5) = 60$.

11. The midpoints of the sides of a triangle are $(7, 4)$, $(1, 2)$, and $(1, 6)$. What is the area of this triangle?

(A) 12 **(B)** 24 **(C)** 30 **(D)** 36 **(E)** 48

Solution: (E) The area of the triangle is 4 times the area of the triangle formed by its midpoints. The base of the midpoint triangle is 4 and its height is 6, giving an area of 12. The original triangle then has area 48.

12. On co-planar lines l and m we choose points P_1, P_2, P_3, P_4, and P_5 on the former; and points Q_1, Q_2, Q_3, Q_4 on the latter. Draw all possible segments with one endpoint one of the P's and the other one of the Q's. What is the maximum total number of points that can be formed by intersection of pairs of these segments?

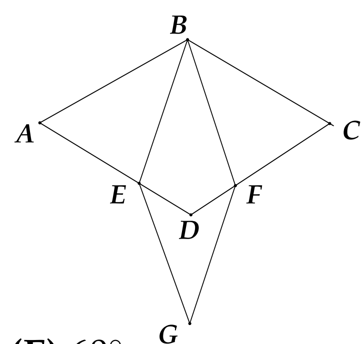
(A) 75 **(B)** 60 **(C)** 45 **(D)** 30 **(E)** 20

Solution: (B) With the lines and points in general position, choosing any two of the P's and any two of the Q's leads to exactly one intersection point and vice-versa. Thus the number of intersection points equals $\binom{5}{2} \cdot \binom{4}{2} = 10 \cdot 6 = 60$.

13. The set S contains seven numbers whose mean is 202. The mean of the four smallest numbers in S equals 100, while the mean of the four largest numbers in S equals 300. What is the median of all the numbers in S?

(A) 184 **(B)** 186 **(C)** 192 **(D)** 196 **(E)** 200

Solution: (B) The sum of all the numbers is 1414; of the four smallest is 400; of the four largest is 1200. Adding the last two numbers exceeds 1414 by 186. This excess must be the median of all seven as it was double counted.



14. Quadrilaterals $ABCD$ and $BEGF$ are rhombi and are situated as in the diagram. If $\angle EBF = 20^\circ$ and $\angle A = 50^\circ$, what is $\angle DEG$?

(A) 40° **(B)** 45° **(C)** 50° **(D)** 55° **(E)** 60°

Solution: (D) Note $\angle G = 20$ so $\angle FEG = 80$. Now $\angle D = 130$, so $\angle DEF = 25$. Thus $\angle DEG = 80 - 25 = 55$.

15. While Xerxes marched on Greece his army streched out for 50 miles. A dispatch rider had to ride from the rear to the head of the army, then

instantly turn about and return to the rear. While he did this, the army advanced 50 miles. How many miles did the rider ride?

(A) 100 **(B)** $50 + 50\sqrt{2}$ **(C)** $100\sqrt{2}$ **(D)** 150 **(E)** $50 + 100\sqrt{2}$

Solution: (B) Let x be the distance the army advanced while the rider went to the front. Then $(50 + x)/x = x/(50 - x)$, since distance is proportional to speed in over equal time intervals. Thus $x = \sqrt{1250} = 25\sqrt{2}$. So the rider travelled $50 + 2x = 50 + 50\sqrt{2}$ miles.

16. How many non-congruent scalene triangles with integer side lengths exist with two sides of lengths 13 and 7 respectively?

    (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

    Solution: (B) The third side may have lengths equal to:
    8,9,10,11,12,14,15,16,17,18, and 19, making eleven such triangles. By the Triangle Inequality and the scalene condition no other possibilities exist.

17. An isosceles trapezoid has bases of 11 and 21 units and legs of 13 units. What is the area of the trapezoid?

    (A) 144 **(B)** 160 **(C)** 176 **(D)** 192 **(E)** 208

    Solution: (D) The trapezoid is the union of two 5-12-13 triangles and an 11 by 12 rectangle and so has the same area as a 16 by 12 rectangle.

18. Hezy and Zeke were employed at different daily wages. At the end of a certain number of days Hezy received \$300, while Zeke, who had been absent from work two of those days, received only \$192. However, had it been the other way around, had Zeke worked all those days and Hezy been absent twice, then both would have received the same amount. What was Hezy's daily wage?

    (A) 30 **(B)** 40 **(C)** 50 **(D)** 60 **(E)** 70

    Solution: (A) Let h and z be Hezy's and Zeke'e daily wages respectively. Let n be the number of days worked. The three conditions given translate into the system:

$$\begin{aligned} hn &= 300 \\ z(n-2) &= 192 \\ zn &= h(n-2) \end{aligned}$$

These lead to $300 - 2h - 2z = 192$ or $h + z = 54$. Letting $z = 54 - h$, we obtain $27n + h = 300$. But $n = 300/h$ which put in the previous equation yields the equivalent quadratic $h^2 - 300h + 8100 = (h - 30)(h - 270) = 0$. Recalling that $h + z = 54$ the correct root is $h = 30$. So Hezy is paid a daily wage of \$30.

19. Suppose that the two legs of a certain right triangle are in the ratio 3 : 4. What is the greatest possible area of such a right triangle, if one of its altitudes measures 24?

    (A) 216 **(B)** 384 **(C)** 486 **(D)** 600 **(E)** 726

    Solution: (D) Since the area of a triangle is one-half the base times the corresponding altitude, then we should choose the altitude to the hypotenuse in order to maximize the area. Let the sides of the triangle be $3x$, $4x$, and $5x$. Then $24 : 3x = 4x : 5x = 4 : 5$, so $x = 10$. Thus the maximum area is $(1/2)(50)(24) = 600$.

20. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

    (A) 32 **(B)** 28 **(C)** 24 **(D)** 20 **(E)** 16

    Solution: (A) Note that by symmetry there are an equal number of either foot. There are 16 that start on the increasing foot. They are in order: 13254, 14253, 14352, 15243, 15342, 24153, 24351, 25143, 25341, 34152, 34251, 35142, 35241, 35412, 45132, and 45231. What 1-1 correspondence makes this symmetry explicit?

Fall 2010 McNabb GDCTM Contest
Algebra II Solutions

NO Calculators Allowed

1. An automobile goes $y/9$ yards in d seconds. How many feet does it travel in two minutes time?

   (A) $\dfrac{40y}{d}$ **(B)** $\dfrac{40d}{3y}$ **(C)** $\dfrac{3y}{40d}$ **(D)** $120yd$ **(E)** $\dfrac{120y}{d}$

   Solution: (A) In one second the auto goes $y/9d$ yards or $y/3d$ feet. In 120 seconds the auto goes $120 \cdot y/3d$ or $40y/d$ feet.

2. The well-known formula $f = (9/5)c + 32$ relates the temperature f in Fahrenheit to the temperature c in Celcius. For how many values of f satisfying $32 \leq f \leq 212$, will the temperature be an integer in both of these scales?

   (A) 9 **(B)** 10 **(C)** 19 **(D)** 20 **(E)** 21

   Solution: (E) These conditions are satisfied precisely when $c = 0, 5, 10, 15, \cdots, 100$, or 21 times.

3. If $f(\sqrt{x}) = \dfrac{x+3}{13}$ and $g(x^2) = x^4 + 3x^2 - 22$, then find $f(g(4))$.

   (A) 3 **(B)** 4 **(C)** 5 **(D)** 6 **(E)** 7

   Solution: (A) Observe $g(4) = g(2^2) = 2^4 + 3(2^2) - 22 = 16 + 12 - 22 = 6$ so $f(g(4) = f(6) = f(\sqrt{36}) = (36+3)/13 = 3$.

4. A square is inscribed in a right triangle with sides of length 3, 4, and 5, so that one of the sides of the square is contained in the hypotenuse of the right triangle. What is the side length of the square?

   (A) $\dfrac{60}{37}$ **(B)** 2 **(C)** $\dfrac{12}{5}$ **(D)** 3 **(E)** cannot be determined

   Solution: (A) Draw the 3-4-5 triangle with the hypotenuse as base. Then sitting atop the inscribed square is a right triangle with sides $3y$, $4y$, and $5y$, with $5y$ also being the side length of the square. By similar triangles, $(4-4y) : 5y = 5 : 3$. Solving this proportion yields $y = 12/37$ so that $5y = 60/37$.

5. The arithmetic mean of a, b, and c is 7 and the arithmetic mean of a^2, b^2, and c^2 is 55. What is the arithmetic mean of ab, bc, and ac?

(A) 24 **(B)** 31 **(C)** 46 **(D)** 48 **(E)** 92

Solution: (C) Note $ab+bc+ca = (1/2)((a+b+c)^2 - (a^2+b^2+c^2)) = (1/2)(21^2 - 165) = 138$. So the average of ab, bc, and ca is $138/3 = 46$.

6. In how many ways can the the letters in the string $ABECEDA$ be arranged so that the consonants are in alphabetical order?

(A) 90 **(B)** 105 **(C)** 120 **(D)** 180 **(E)** 210

Solution: (E) Given an initial arrangement of the letters, there are six ways to scramble the 3 consonants (including the original arrangement) only one of which has the consonants in alphabetical order. Thus one-sixty of all possible arrangements of these letters is what we seek. Our answer is then $(1/6)7!/(2!2!) = 210$.

7. The graph of the quadratic function $f(x) = ax^2 + bx + c$ contains the points $(-1,6)$, $(7,6)$, and $(1,-6)$. What is the minimum value of $f(x)$?

(A) -36 **(B)** -26 **(C)** -20 **(D)** -10 **(E)** -6

Solution: (D) Since the y coordinates of $(-1,6)$ and $(7,6)$ are equal, the x coordinate of the vertex must be the average of -1 and 7, so 3. Then one form of the equation of the parabola is $y = a(x-3)^2 + k$. Substituting in the points $(7,6)$, and $(-1,6)$ into this equation yields the system $16a + k = 6$ and $4a + k = -6$. Solving for k gives $k = -10$, our minimum value.

8. Let a, b, and c be positive real numbers. Supposing that $ab = kc$, $ac = lb$, and $bc = ma$, then c must equal

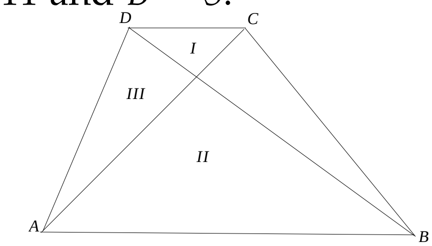
(A) lm **(B)** $\sqrt{klm}$ **(C)** $\sqrt{\frac{l}{m}}$ **(D)** $k\sqrt{lm}$ **(E)** $\sqrt{lm}$

Solution: (E) Multiply all three equations together and take the square root of both sides to yield $abc = lkm$. Replace ab by kc, so $kc^2 = lkm$ or $c^2 = lm$. Thus since $c > 0$, $c = \sqrt{lm}$.

9. The real number $\sqrt{16+\sqrt{220}}$ can be expressed in the form $\sqrt{A}+\sqrt{B}$, where A and B are integers and $A > B$. What is the value of $A - B$?

(A) 6 **(B)** 7 **(C)** 8 **(D)** 9 **(E)** 10

Solution: (A) From $16+\sqrt{220} = A + B + 2\sqrt{AB}$ we see that, as A and B are positive integers, $A + B = 16$ and $AB = 55$. So $A = 11$ and $B = 5$.



10. In trapezoid $ABCD$, the area of region I is 9 and the area of region II is 16. What is the area of region III?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 12.5 **(E)** cannot be determined

Solution: (C) The area of region III is the geometric mean of the areas of regions I and II. Note regions I and II are similar triangles. Let a be the top base of the trapezoid, A =area of region I, b be the bottom base of the trapezoid, B =area of region II, and C = the area of region III. Then $a/b = \sqrt{A}/\sqrt{B}$, $(A+C)/A = (a+b)/a$, and $C/A = b/a$, so $C = (b/a)A = (\sqrt{B}/\sqrt{A})A = \sqrt{AB}$.

11. The polynomial

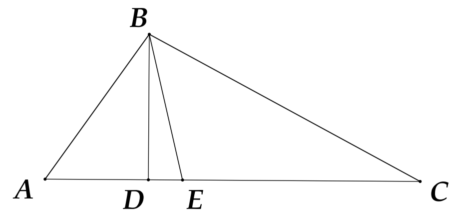
$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

can be factored as $P_2(x) \cdot P_6(x)$ where P_2 and P_6 are polynomials with integer coefficients and have degrees 2 and 6 respectively. Find the sum of the coefficients of P_2.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 9

Solution: (C) The factorization of the given polynomial is gotten by factor by grouping, grouping this first three terms, the second three and the third three: $(x^2 + x + 1)(x^6 + x^3 + 1)$.

12. In $\triangle ABC$, $\angle A = 60°$, $C = 40°$, $BD \perp AC$, and $\overrightarrow{BE}$ bisects $\angle ABC$. Find the measure of $\angle DBE$ in degrees.



(A) 8 **(B)** 10 **(C)** 12 **(D)** 14 **(E)** 20

Solution: (B) Note $\angle B = 80$ so $\angle ABE = 40$. But $\angle ABD = 30$, so that $\angle DBE = 40 - 30 = 10$.

13. A bag contains 4 quarters and 2 dimes. If 3 coins are randomly removed from the bag, what is the expected total value in cents of these three coins?

(A) 50 **(B)** 55 **(C)** 60 **(D)** 65 **(E)** 75

Solution: (C) The probability of drawing 3 quarters is $1/5$; of drawing two quarters is $3/5$; of one quarter is $1/5$. The expected value of the draw is $75(1/5) + 60(3/5) + 45(1/5) = 60$.

14. The set S contains seven numbers whose mean is 202. The mean of the four smallest numbers in S equals 100, while the mean of the four largest numbers in S equals 300. What is the median of all the numbers in S?

(A) 184 **(B)** 186 **(C)** 192 **(D)** 196 **(E)** 200

Solution: (B) The sum of all the numbers is 1414; of the four smallest is 400; of the four largest is 1200. Adding the last two numbers exceeds 1414 by 186. This excess must be the median of all seven as it was double counted.

15. Let a and b be positive constants. If x is a solution of

$$\sqrt{x+a} + \sqrt{x+b} = \sqrt{a+b}$$

then x must equal

(A) 0 **(B)** $\frac{a+b}{2}$ **(C)** $-\frac{ab}{a+b}$

(D) $-\left(\frac{1}{a} + \frac{1}{b}\right)$ **(E)** $-\frac{2}{a+b}$

Solution: (C) Subtract $\sqrt{x+b}$ from both sides and square. Simplify and square again to obtain $b^2 = (a+b)(x+b)$ or $x = -ab/(a+b)$.

16. While Xerxes marched on Greece his army streched out for 50 miles. A dispatch rider had to ride from the rear to the head of the army, then instantly turn about and return to the rear. While he did this, the army advanced 50 miles. How many miles did the rider ride?

(A) 100 **(B)** $50 + 50\sqrt{2}$ **(C)** $100\sqrt{2}$
(D) 150 **(E)** $50 + 100\sqrt{2}$

Solution: (B) Let x be the distance the army advanced while the rider went to the front. Then $(50 + x)/x = x/(50 - x)$, since distance is proportional to speed in over equal time intervals. Thus $x = \sqrt{1250} = 25\sqrt{2}$. So the rider travelled $50 + 2x = 50 + 50\sqrt{2}$ miles.

17. The sum of two of the roots of $p(x) = 4x^3 + 8x^2 - 9x - k$, where k is a constant, is zero. Find the value of k.

(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 200

Solution: (D) Let the roots be r, $-r$, and s. Then $r + (-r) + s = -8/4 = -2$, so $s = -2$. Then $-r^2 + 2r - 2r = -9/4$ so $r = \pm 3/2$. Then $-k/4 = -(3/2)(-3/2)(-2)$ and $k = 18$.

18. Let f be a function such that $f(x + y) = f(xy)$ for all real numbers x and y. If it is also known that $f(5) = 5$, determine the value of $f(25)$.

(A) 1 **(B)** 5 **(C)** 10 **(D)** 20 **(E)** 25

Solution: (B) Let $y = 0$, so $f(x) = f(0)$ for all x. So $f(5) = f(0) = 5 = f(x)$ for all x. Then $f(25) = 5$.

19. Hezy and Zeke were employed at different daily wages. At the end of a certain number of days Hezy received \$300, while Zeke, who had been absent from work two of those days, received only \$192. However, had it been the other way around, had Zeke worked all those days and Hezy been absent twice, then both would have received the same amount. What was Hezy's daily wage?

(A) 30 **(B)** 40 **(C)** 50 **(D)** 60 **(E)** 70

Solution: (A) Let h and z be Hezy's and Zeke'e daily wages respectively. Let n be the number of days worked. The three conditions given translate into the system:

$$\begin{aligned} hn &= 300 \\ z(n-2) &= 192 \\ zn &= h(n-2) \end{aligned}$$

These lead to $300 - 2h - 2z = 192$ or $h + z = 54$. Letting $z = 54 - h$, we obtain $27n + h = 300$. But $n = 300/h$ which put in the previous equation yields the equivalent quadratic $h^2 - 300h + 8100 = (h-30)(h-270) = 0$. Recalling that $h + z = 54$ the correct root is $h = 30$. So Hezy is paid a daily wage of \$30.

20. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

    (A) 32 **(B)** 28 **(C)** 24 **(D)** 20 **(E)** 16

    Solution: (A) Note that by symmetry there are an equal number of either foot. There are 16 that start on the increasing foot. They are in order: 13254, 14253, 14352, 15243, 15342, 24153, 24351, 25143, 25341, 34152, 34251, 35142, 35241, 35412, 45132, and 45231. What 1-1 correspondence makes this symmetry explicit?

Fall 2010 McNabb GDCTM Contest
Calculus Solutions

NO Calculators Allowed

1. How many non-congruent scalene triangles with integer side lengths exist with two sides of lengths 13 and 7 respectively?

   (A) 10 **(B)** 11 **(C)** 12 **(D)** 13 **(E)** 14

   Solution: (B) The third side may have lengths equal to: 8,9,10,11,12,14,15,16,17,18, and 19, making eleven such triangles. By the Triangle Inequality and the scalene condition no other possibilities exist.

2. Let A and B satisfy $\log_2(\log_4 A) = 1$ and $\log_4(\log_2 B) = 1$. Find the value of $\frac{1}{A} + \frac{1}{B}$.

   (A) 2 **(B)** 1 **(C)** $\frac{1}{3}$ **(D)** $\frac{1}{8}$ **(E)** $\frac{1}{16}$

   Solution: (D) It is straightforward to verify that $A = B = 16$.

3. The value of $\sin 195^\circ + \sin 105^\circ$ is equal to

   (A) $\frac{1}{2}$ **(B)** $\frac{1}{\sqrt{2}}$ **(C)** $\frac{\sqrt{3}}{2}$ **(D)** 1 **(E)** $\sqrt{2}$

   Solution: (B) Let $Q = \sin 195 + \sin 105$. Note that $Q = \sin 75 - \cos 75$. From $\sin 30 = \sin 75 \cos 45 - \cos 75 \sin 45 = Q/\sqrt{2} = 1/2$, it follows that $Q = 1/\sqrt{2}$.

4. A positive integer has the interesting property that when expressed as a three digit base-7 number, those digits are the reverse of its digits when expressed as a base-9 number. What is this number expressed in normal form as a base-10 number?

   (A) 124 **(B)** 241 **(C)** 248 **(D)** 428 **(E)** 503

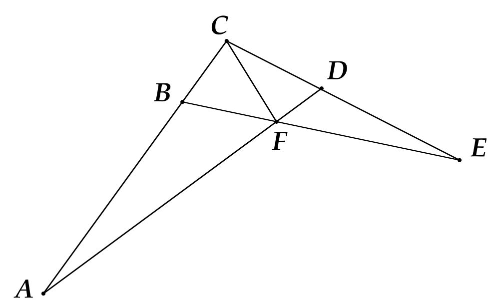
Solution: (C) Let $\underline{ABC}$ be the base-7 digits of the integer sought. Then $49A + 7B + C = 81C + 9B + A$ or $B = 24A - 40C$. The digit B must be drawn from the digits 0 through 6. When A is 2 and C is 1, B is too large. The next possible answer occurs when A is 5 and C is 3. This would force B to equal 0 and so works. No other combination works. So the number is base-7 is 503. In base-10 this comes to 248.

5. In how many ways can the the letters in the string $ABECEDA$ be arranged so that the consonants are in alphabetical order?

(A) 90 **(B)** 105 **(C)** 120 **(D)** 180 **(E)** 210

Solution: (E) Given an initial arrangement of the letters, there are six ways to scramble the 3 consonants (including the original arrangement) only one of which has the consonants in alphabetical order. Thus one-sixty of all possible arrangements of these letters is what we seek. Our answer is then $(1/6)7!/(2!2!) = 210$.

6. In the configuration shown, the area of $\triangle ABF$ is 11, the area of $\triangle CFD$ is 3, and the area of $\triangle DEF$ is 11/3. Find the area of $\triangle BCF$.



(A) $\dfrac{11}{4}$ **(B)** 3 **(C)** 4 **(D)** $\dfrac{14}{3}$ **(E)** 8

Solution: (C) Rescale $\triangle BCE$ so that it has area equal to that of $\triangle CAD$. If x is the area of triangle BCF then this rescaling leads to

$$(11/3 + 3 + x)(9/20)(11/x + 1) = x + 3 + 11$$

This equation reduces to $x^2 + 11x - 60 = (x - 4)(x + 15) = 0$. Since $x > 0$ then $x = 4$.

7. Hezy and Zeke were employed at different daily wages. At the end of a certain number of days Hezy received \$300, while Zeke, who had been absent from work two of those days, received only \$192. However, had it been the other way around, had Zeke worked all those days and Hezy been absent twice, then both would have received the same amount. What was Hezy's daily wage?

(A) 30 **(B)** 40 **(C)** 50 **(D)** 60 **(E)** 70

Solution: (A) Let h and z be Hezy's and Zeke'e daily wages respectively. Let n be the number of days worked. The three conditions given translate into the system:

$$\begin{aligned} hn &= 300 \\ z(n-2) &= 192 \\ zn &= h(n-2) \end{aligned}$$

These lead to $300 - 2h - 2z = 192$ or $h + z = 54$. Letting $z = 54 - h$, we obtain $27n + h = 300$. But $n = 300/h$ which put in the previous equation yields the equivalent quadratic $h^2 - 300h + 8100 = (h-30)(h-270) = 0$. Recalling that $h + z = 54$ the correct root is $h = 30$. So Hezy is paid a daily wage of \$30.

8. What is the slope of the line that bisects the acute angle formed by the lines $y = (5/12)x$ and $y = (3/4)x$?

(A) $\frac{1}{2}$ **(B)** $\frac{7}{12}$ **(C)** $\frac{5}{8}$ **(D)** $\frac{4}{7}$ **(E)** $\frac{2}{3}$

Solution: (D) Let α be the angle in a 5-12-13 triangle opposite the side of length 5; let β be the angle in a 3-4-5 triangle opposite the side of length 3. Then the slope of the line we seek is equal to

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\frac{5}{13}\cdot\frac{4}{5} + \frac{12}{13}\cdot\frac{2}{5}}{1 + \frac{12}{13}\cdot\frac{4}{5} - \frac{5}{13}\cdot\frac{3}{5}} = \frac{4}{7}$$

9. The sum of two of the roots of $p(x) = 4x^3 + 8x^2 - 9x - k$, where k is a constant, is zero. Find the value of k.

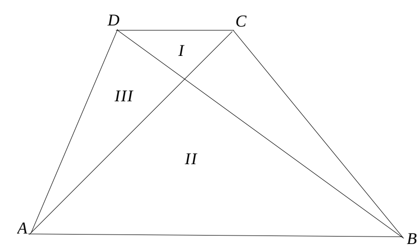
(A) 3 **(B)** 6 **(C)** 12 **(D)** 18 **(E)** 200

Solution: (D) Let the roots be r, $-r$, and s. Then $r + (-r) + s = -8/4 = -2$, so $s = -2$. Then $-r^2 + 2r - 2r = -9/4$ so $r = \pm 3/2$. Then $-k/4 = -(3/2)(-3/2)(-2)$ and $k = 18$.

10. An urn contains two red, two blue, two white, and two yellow balls. Susan draws balls at random from the urn without replacing them. What is the expected number of draws Susan makes until drawing her first red ball?

(A) $\frac{42}{14}$ **(B)** $\frac{43}{14}$ **(C)** $\frac{42}{13}$ **(D)** $\frac{43}{13}$ **(E)** $\frac{44}{13}$

Solution: (A) The expected value is $1 \cdot P_1 + 2 \cdot P_2 + \cdots + 7 \cdot P_7$, where P_k =the probability that the first red ball is drawn on the kth draw. A careful calculation shows that this sum is $1/4 + 3/7 + 15/28 + 4/7 + 15/28 + 3/7 + 1/4 = 42/14$.



11. In trapezoid $ABCD$, the area of region I is 9 and the area of region II is 16. What is the area of region III?

(A) 10 **(B)** 11 **(C)** 12 **(D)** 12.5 **(E)** cannot be determined

Solution: (C) The area of region III is the geometric mean of the areas of regions I and II. Note regions I and II are similar triangles. Let a be the top base of the trapezoid, A =area of region I, b be the bottom base of the trapezoid, B =area of region II, and C = the area of region III. Then $a/b = \sqrt{A}/\sqrt{B}$, $(A + C)/A = (a + b)/a$, and $C/A = b/a$, so $C = (b/a)A = (\sqrt{B}/\sqrt{A})A = \sqrt{AB}$.

12. The polynomial

$$x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$$

can be factored as $P_2(x) \cdot P_6(x)$ where P_2 and P_6 are polynomials with integer coefficients and have degrees 2 and 6 respectively. Find the sum of the coefficients of P_2.

(A) 1 **(B)** 2 **(C)** 3 **(D)** 6 **(E)** 9

Solution: (C) The factorization of the given polynomial is gotten by factor by grouping, grouping this first three terms, the second three and the third three: $(x^2 + x + 1)(x^6 + x^3 + 1)$.

13. The number of ordered pairs of integers that satisfy the equation $x^2 + 4x + y^2 = 9$ is

(A) 0 **(B)** 2 **(C)** 4 **(D)** 6 **(E)** 8

Solution: (E) The circle may be translated by whole integer horizontal displacement to the circle $x^2 + y^2 = 13$. The only fundamental solution to this equation is $2^2 + 3^2 = 13$. By swapping x and y and exhausting all $\pm$ variation, we find all 8 solutions.

14. Let a and b be positive constants. If x is a solution of

$$\sqrt{x+a} + \sqrt{x+b} = \sqrt{a+b}$$

then x must equal

(A) 0 **(B)** $\dfrac{a+b}{2}$ **(C)** $-\dfrac{ab}{a+b}$ **(D)** $-\left(\dfrac{1}{a} + \dfrac{1}{b}\right)$ **(E)** $-\dfrac{2}{a+b}$

Solution: (C) Subtract $\sqrt{x+b}$ from both sides and square. Simplify and square again to obtain $b^2 = (a+b)(x+b)$ or $x = -ab/(a+b)$.

15. Which of the following lines is an asymptote of the curve $x^2 - 4xy + 3y^2 = 7$?

(A) $x + 3y = 0$ **(B)** $x - 3y = 0$ **(C)** $x + y = 0$ **(D)** $x + y = -3$ **(E)** $x - y = 1$

Solution: (B) Replace y by $mx + b$ in the equation of the curve and set the coefficients of the two highest powers of x equal to zero. This guarantees intersection and tangency of the curve and the asymptote at infinity. Carrying this out gives $3m^2 - 4m + 1 = (3m-1)(m-1) = 0$ and $2b(3m-2) = 0$. So $m = 1/3$ or $m = 1$, and in either case, $b = 0$. Thus $x - 3y = 0$ is an asympote while the other choices are not.

16. A five-digit integer, with all distinct digits which in this problem must be 1,2,3,4, and 5 in some order, is called *alternating* if the digits alternate between increasing and decreasing in size as read from left to right. They may start on an increasing or decreasing foot. For instance, both 34152 and 53412 are alternating while 12354 is not, for example. How many of this kind of 5 digit integer are alternating?

    (A) 32 **(B)** 28 **(C)** 24 **(D)** 20 **(E)** 16

    Solution: (A) Note that by symmetry there are an equal number of either foot. There are 16 that start on the increasing foot. They are in order: 13254, 14253, 14352, 15243, 15342, 24153, 24351, 25143, 25341, 34152, 34251, 35142, 35241, 35412, 45132, and 45231. What 1-1 correspondence makes this symmetry explicit?

17. Evaluate the following limit:

$$\lim_{x\to\infty} \left(x^3 + ax^2\right)^{1/3} - \left(x^3 - ax^2\right)^{1/3}$$

    (A) 0 **(B)** 1 **(C)** $\frac{a}{3}$ **(D)** $\frac{2a}{3}$ **(E)** a

    Solution: (D) Let $u^3 = x^3 + ax^2$ and $v^3 = x^3 - ax^2$. Multiply and divide by the cubic conjugate $u^2 + uv + v^2$. The numerator reduces to $2ax^2$. Divide numerator and denominator by x^2 and take the limit to find the result of $2a/3$.

18. In the piecewise function $f(x)$ given by

$$f(x) = \begin{cases} \dfrac{1}{1+x^2} & \text{if } x \le 1 \\ a(x-1)^2 + b(x-1) + c & \text{if } x > 1 \end{cases}$$

the constants a, b, and c are chosen to ensure that f is twice differentiable over the real numbers. What then must be the value of a?

(A) -2 **(B)** -1 **(C)** $-\frac{1}{2}$ **(D)** $\frac{1}{4}$ **(E)** $\frac{1}{2}$

Solution: (D) The second derivative at $x = 1$ must match on both sides. On the right hand side the second derivative is $2a$ while on the left hand side it is the value of $(6x^2 - 2)/(1 + x^2)^3$ at $x = 1$, i.e., $1/2$. So $a = 1/4$.

19. Let $f(x)$ be differentiable with $f' > 0$. Let $g = f^{-1}$. Find the value of $(g \circ g)'(3)$ if $f(4) = 3$, $f'(4) = 2$, $f(5) = 4$, and $f'(5) = 5$.

(A) $\frac{1}{10}$ **(B)** $\frac{1}{5}$ **(C)** 2 **(D)** 5 **(E)** 20

Solution: (A) From the chain rule, the inverse function rule, and the given data, $(g \circ g)'(3) = g'(4)g'(3) = (1/5)(1/2) = 1/10$.

20. Let g be 10 times differentiable with $g(0) = 1/8!$. Suppose $f(x) = x^{10}g(x)$. Find $f^{(10)}(0)$.

(A) $\frac{1}{8!}$ **(B)** 1 **(C)** 90 **(D)** 120 **(E)** 8!

Solution: (C) Imagine carrying out the product rule ten times. Now observe that when substituting zero for x in that tenth derivative any term still having an x factor will contribute nothing to the overall sum. The only term without any factor of x is the term that results from putting all ten derivatives on x^{10}. This term is $10!g(x)$, which at 0 yields $10!/8! = 90$.