Spring 2018 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed/ 45 Minutes

1. Find the smallest integer n so that 2^n is greater than 100.

2. Hezzy runs his first lap in 80 seconds, and, since he gets more and more tired as he runs, each following lap takes him six seconds longer than the previous one. If five laps of this track equals a mile, then Hezzy will take how many seconds to run a mile?

3. A certain triangle has an area of 200. If its height is doubled but its base kept the same, what is the area of the new triangle?

4. The positive integers are put in a rectangular grid in the following way

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

   What is the number at the top of the column which contains the number 70?

5. Bitecoin was created in 2004. By 2011 its value had increased 550% from its orginal value. By 2018 its value had increasesd 2100% from its value in 2011. If you had invested $\$10,000$ in Bitecoin in 2004, what would your investment be worth now in 2018?

6. The points A, B, and C are collinear with $AB = 18$ and $BC = 32$. Find the sum of all possible values of AC.

7. How many positive factors of $10,000$ are greater than or equal to 100?

8. In how many ways can change be made for two dollars if you have an unlimited number of nickels, dimes, and quarters available?

9. Joe is 5/4 as tall as Susan. Bob is 3/4 as tall as Max. Susan is three inches shorter than Bob. If Bob is 51 inches tall, then how much taller is Max than Joe? Answer in inches.

10. When Billy starts his trip Sam has a seventy-five mile head start. If Billy drives at a rate of 70 miles per hour and catches up to Sam in two-and-a-half hours, find Sam's driving rate in miles per hour.

11. How many six-digit numbers of the form

$$456AB1$$

   are multiples of seven? Note A and B are digits.

12. John has 43 blue socks, 71 red socks, and 61 green socks. If John were to select from his collection of socks at random, what is the fewest number of socks he would need to draw in order to guarantee fifteen matching pairs?

13. What is the area of a triangle with side lengths 6, 6, and 4?

14. Find the sum of the 100 smallest positive integers that are neither multiples of two nor multiples of three.

15. How many pairs of ordered integers (m, n) satisfy $m \geq 0$, $n \geq 0$, and

$$\frac{m}{144} + \frac{n}{108} \leq 1$$

?

Spring 2018 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed/ 45 Minutes

1. A train passes a standing observer in seven seconds and a 378 yard platform station in 25 seconds. How long in yards is the train?

2. In how many ways can the letters in MCNABB be arranged such that the M and the N are never adjacent but the two B's are always adjacent?

3. A solution of 18 liters is 20% acid. How many liters of 68% acid must be added to the original solution to form a solution that is 36% acid?

4. In how many ways can 80 identical marbles be divided among eight children so that two of the children get 16 marbles each, four of them get 10 each, and two of them get 4 each?

5. How many integers n satisfy
$$\frac{15n}{13} - 8 < \frac{11n}{9} < \frac{13n}{11} + 8$$

6. Let a be a fixed positive real number. Find the area of the triangle formed by the three lines
$$\begin{aligned} y &= ax \\ y &= \frac{x}{a} \\ x + y &= a \end{aligned}$$
in terms of a.

7. Events A, B, and C are mutually independent with $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.15$, find the probability of the event $A \cup B \cup C$.

8. A thin metal plate of uniform density has the shape of a quadrilateral with vertices at $(0, 0)$, $(6, 0)$, $(4, 4)$, and $(2, 4)$. Find the coordinates of the center of mass of this plate.

9. A biased coin is such that when tossed eight times the probability of getting exactly three heads is the same as the probability of getting exactly four heads. When this coin is tossed once, what is the probability of getting heads? Assume that the probability of getting heads when tossed once is neither zero nor one.

10. What is the fewest number of multiplications required to calculate a^{45}? The least efficient way is to multiply one a at a time for a total of 44 multiplications. A more efficient way is to multiply one a at a time to get to a^{15}. Then multiply three a^{15}'s together. This requires 16 multiplications altogether.

11. If r and s are the roots of the quadratic equation
$$2x^2 + 2x - 17 = 0$$
find the value of
$$r^2s^2 + rs^2 + sr^2 + rs + 1$$

12. After expanding and simplifying the product

$$a(a+b)(a+b+c)(a+b+c+d)(a+b+c+d+e)(a+b+c+d+e+f)$$

how many terms remain?

13. Solve:

$$|x-5|-|3x+5|+|2x+10|=4$$

14. Find one positive integer value of n so that 12^n-1 is divisible by 25.

15. For what value of the positive parameter a does the triangle with vertices $(0,0)$, $(a,2\sqrt{a})$, and $(-2/a,1/\sqrt{a})$ have the least possible area?

Spring 2018 McNabb GDCTM Contest
Geometry

NO Calculators Allowed/ 60 Minutes

1. A certain triangle has an area of 200. If its height is doubled but its base kept the same, what is the area of the new triangle?

2. The points A, B, and C are collinear with $AB = 18$ and $BC = 32$. Find the sum of all possible values of AC.

3. What is the area of a triangle with side lengths 6, 6, and 4?

4. The positive integers are put in a rectangular grid in the following way

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

   What is the number at the top of the column which contains the number 70?

5. Quadrilateral $ABCD$ has $AB = BC = 15$, $CD = 12$, and $DA = 9$. If $\angle B = 60°$, find the area of $ABCD$.

6. A thin metal plate of uniform density has the shape of a quadrilateral with vertices at $(0, 0)$, $(6, 0)$, $(4, 4)$, and $(2, 4)$. Find the coordinates of the center of mass of this plate.

7. Solve:

$$|x - 5| - |3x + 5| + |2x + 10| = 4$$

8. Events A, B, and C are mutually independent with $P(A) = 0.2$, $P(B) = 0.3$, and $P(C) = 0.15$, find the probability of the event $A \cup B \cup C$.

9. Let a be a fixed positive real number. Find the area of the triangle formed by the three lines

$$\begin{aligned} y &= ax \\ y &= \frac{x}{a} \\ x + y &= a \end{aligned}$$

   in terms of a.

10. For what value of the positive parameter a does the triangle with vertices $(0, 0)$, $(a, 2\sqrt{a})$, and $(-2/a, 1/\sqrt{a})$ have the least possible area?

11. Three different colors are available to color the sides of a square. In how many different ways can this be done? Two ways are the same if one can be rotated into the other.

12. In trapezoid $ABCD$ with $AB \parallel CD$ and $AB/CD = 1/6$, draw diagonals AC and BD intersecting at point E. Find the ratio of the area of $ABCD$ to the area of ABE.

13. Factor $x^4 - 6x^3 + 9x^2 - 4$ into two quadratic polynomials with integer coefficients.

14. Draw the circle inscribed in the triangle with sides 3, 4, and 5. Then draw a second circle that is externally tangent to the first circle and is also tangent to the sides of length 3 and 5. Find the radius of this second circle.

15. In $\triangle ABC$, the angle bisectors of $\angle B$ and $\angle C$ meet at point Q. The line through Q parallel to AB meets side BC at S. The line through Q parallel to AC meets side BC at T. The line through Q parallel to BC meets sides AB and AC at points P and R respectively. If $AB = 4$, $BC = 9$, and $CA = 11$, find the ratio ST/PR.

Spring 2018 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed/ 60 Minutes

1. The positive integers are put in a rectangular grid in the following way

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 18 & 17 & 16 & 15 & 14 & 13 & 12 & 11 & 10 \\ 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

   What is the number at the top of the column which contains the number 70?

2. A solution of 18 liters is 20% acid. How many liters of 68% acid must be added to the original solution to form a solution that is 36% acid?

3. Find the maximum possible value of $a + b + c$ if a, b, and c must satsify

$$\begin{aligned} 4a + b + 9c &\leq 13 \\ 2a + 18b + 7c &\leq 22 \\ 21a + 8b + 11c &\leq -15 \end{aligned}$$

4. A train passes a standing observer in seven seconds and a 378 yard platform station in 25 seconds. How long in yards is the train?

5. Let a be a fixed positive real number. Find the area of the triangle formed by the three lines

$$\begin{aligned} y &= ax \\ y &= \frac{x}{a} \\ x + y &= a \end{aligned}$$

   in terms of a.

6. In trapezoid $ABCD$ with $AB \parallel CD$ and $AB/CD = 1/6$, draw diagonals AC and BD intersecting at point E. Find the ratio of the area of $ABCD$ to the area of ABE.

7. How many integers n satisfy
$$\frac{15n}{13} - 8 < \frac{11n}{9} < \frac{13n}{11} + 8$$

8. One counter is in a pouch. It is with equal probability either black or white. A white counter is added to the pouch. Next, one counter is randomly drawn from the pouch and it turns out to be white. What is the probability that the remaining counter in the pouch is black?

9. Find the period of the function

$$f(x) = \cos(7x/3) + \sin(3x/7)$$

   where x is measured in radians.

10. Find the sum

$$i + i^2 + i^3 + i^4 + i^5 + i^6 + i^8 + i^9 + i^{10} + i^{11} + i^{12} + i^{13} + i^{15} + \cdots + i^{139} + i^{141}$$

where all the powers of i that have exponents that are multiples of seven have been omitted. Here i is a square root of -1.

11. Factor $x^4 - 6x^3 + 9x^2 - 4$ into two quadratic polynomials with integer coefficients.

12. Let r and s be the roots of the monic quadratic $x^2 + 3x - 7$. If the monic quadratic $x^2 + bx + c$ has roots $r + 2$ and $s + 2$, find the value of bc.

13. A peculiar cat climbs stairs by taking steps only two or three at a time. Bored one day, the curious cat climbs over and over again the same stair, and finds that it can climb this set of stairs in exactly 114 different ways. How many steps does the staircase have?

14. Find the remainder when the polynomial

$$x^{30} + x^{24} + x^{18} + x^{12} + x^6 + 1$$

is divided by $x^5 + x^4 + x^3 + x^2 + x + 1$.

15. Integers a and b have the property that the cubic equations

$$x^3 + 10x^2 - 16x + a = 0$$
$$x^3 + 18x^2 + 88x + b = 0$$

share exactly two real roots. Find the value of $a + b$.

Spring 2018 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed/ 60 Minutes

1. Hezzy runs his first lap in 80 seconds, and, since he gets more and more tired as he runs, each following lap takes him six seconds longer than the previous one. If five laps of this track equals a mile, then Hezzy will take how many seconds to run a mile?

2. One counter is in a pouch. It is with equal probability either black or white. A white counter is added to the pouch. Next, one counter is randomly drawn from the pouch and it turns out to be white. What is the probability that the remaining counter in the pouch is black?

3. Let x and y satisfy the system

$$\tan x + \tan y = 20$$
$$\cot x + \cot y = 15$$

   Find the value of $\tan(x+y)$.

4. If r and s are the roots of the quadratic equation

$$2x^2 + 2x - 17 = 0$$

   find the value of

$$r^2s^2 + rs^2 + sr^2 + rs + 1$$

5. Solve for a:

$$\begin{vmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{vmatrix} = 0$$

   Note that the vertical bars here indicate the determinant is to be taken.

6. How many pairs of ordered integers (m, n) satisfy $m \geq 0$, $n \geq 0$, and

$$\frac{m}{144} + \frac{n}{108} \leq 1$$

   ?

7. A biased coin is such that when tossed eight times the probability of getting exactly three heads is the same as the probability of getting exactly four heads. When this coin is tossed once, what is the probability of getting heads? Assume that the probability of getting heads when tossed once is neither zero nor one.

8. Find one positive integer value of n so that $12^n - 1$ is divisible by 25.

9. Find all ordered triples (x, y, z) that solve the system

$$xy = 5x + 6y - 4z$$
$$y^2 = 3x + 5y - z$$
$$yz = x + 4y + 2z$$

10. A fair coin is tossed 13 times. Find the expected number of HH pairs. We count HH pairs thus—the sequence

$$HTTHHTHHHTTHH$$

has four HH pairs.

11. How many 3×3 matrices with non-negative integer entries have each row sum and each column sum equal to 2?

12. How many ordered pairs of integers (x, y) satisfy

$$x + y = x^2 - xy + y^2$$

?

13. After expanding and simplifying the product

$$a(a+b)(a+b+c)(a+b+c+d)(a+b+c+d+e)\cdots(a+b+c+\cdots+m)$$

how many terms remain?

14. Integers a and b have the property that the cubic equations

$$x^3 + 10x^2 - 16x + a = 0$$
$$x^3 + 18x^2 + 88x + b = 0$$

share exactly two real roots. Find the value of $a + b$.

15. Recall that $\binom{n}{k}$ is the number of ways of choosing k objects from a set of n objects. Find the value of

$$\binom{2019}{0} - \binom{2018}{1} + \binom{2017}{2} - \binom{2016}{3} + \binom{2015}{4} - \cdots + \binom{1011}{1008} - \binom{1010}{1009}$$

Spring 2018 McNabb GDCTM Contest
Calculus

NO Calculators Allowed/ 60 Minutes

Assume all variables are real unless otherwise stated in the problem.

1. A tennis tournament has ten players registered to play singles. How many different first-round pairings are possible? In the first-round there are five matches and all ten players play.

2. Find the sum of the coefficients of all the even powers of x when
$$(x^5 - x^2 + 1)^{10}$$
is expanded and simplified.

3. In how many ways can you choose five of these letters
$$a, b, c, d, e, f, g, h, i, j, k, x, x, x, x, x$$
?

4. Find $\frac{d^{10}y}{dx^{10}}$ if $y = e^{-x} \cos x$.

5. For which values of the parameter a are all the roots of the polynomial $x^4 + ax^2 + 1$ real?

6. Find the absolute maximum value of the function $f(\theta) = \cos^3 \theta \sin \theta$ over the interval $[0, \pi/2]$.

7. There exists a function continuous at each real number but differentiable at no real number. Answer True or False.

8. For what values of the real parameter a is the function
$$f(x) = x^4 + ax^3 + ax^2 + ax + a$$
concave up on the entire real number line? Answer in interval notation.

9. Find
$$\lim_{x \to \infty} \sqrt{x + \sqrt{x}} - \sqrt{x}$$

10. Find the smallest possible value of the the constant m such that the inequality $mx - 1 + 1/x \geq 0$ holds for all $x > 0$.

11. Find the maximum value of the expression
$$16x^5 - 20x^3 + 5x$$
as x varies over the interval $-1 \leq x \leq 1$.

12. A certain radioactive isotope has a mean time to decay of 12 seconds. Given a very large number of such atoms, how many seconds do you have to wait until half of them have decayed?

13. Find the first four non-zero terms of the Taylor polynomial approximation centered at zero of the solution of the differential equation

$$\frac{dy}{dx} = xy + 1$$

with initial condition $y(0) = 3$.

14. Find a function $f(t)$ that satisfies for all x

$$16x^2 + 16x + 4 = \int_{3x+1}^{5x+2} f(t)\, dt$$

15. Find the maximum possible value of $2a + b$ if $a \geq 0$, $b \geq 0$, and $8a^3 = 8ab - b^3$.