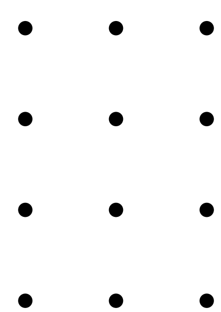
Fall 2016 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed

1. How many two-digit integers are divisible by five?
2. Simplify: $7 - 5 \cdot 13$
3. Jerry has five times as many comic books as Tom. If Jerry has forty-five comic books, how many does Tom have?
4. A recipe for 5 servings calls for $2\frac{1}{12}$ cups of flour. To adjust this recipe to serve a dozen, how many cups of flour should now be used?
5. Jane was born in the year 2003. When she was born, her Mom was 26 years old. In what year will Jane's Mom be three times older than her?
6. Find the number of squares whose vertices all are points of a uniform 3 by 4 rectangular array, as shown below:



7. Express the number 2016_8 (means 2016 base 8) in base 9.
8. The volume of a rectangular box is 360. If the height of the box is increased by 3%, the width of the box decreased by 5% and the depth of the box increased by 2%, then find the volume of the new box.
9. Hezy and Zeke have between them $30. If Zeke were to give Hezy four dollars, they would then have the same amount of money. How many dollars did Zeke have to begin with?
10. How many zero's occur when the number $2^{23} * 3 * 5^{24} * 7$ is written out in standard form?
11. If the sum of three positive integers is 32, what is the greatest possible value of their product?

12. Simplify

$$\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3+\cfrac{1}{4}}}}$$

13. If three standard dice are rolled, what is the probability that all three show a different number?

14. Which of these numbers is the least?

$$\{2-\sqrt{3},\ 1/3,\ (.57)^2\}$$

15. If the $\text{lcm}(80, a) = 320$, how many different positive integer values can a take on?

Fall 2016 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. Jerry has five times as many comic books as Tom. If Jerry has forty-five comic books, how many does Tom have?
2. Jane was born in the year 2003. When she was born, her Mom was 26 years old. In what year will Jane's Mom be three times older than her?
3. Simplify
$$5x^2 - 7x + 3 - 3(x+4)^2$$
4. John has some nickels and quarters, 37 coins in all. If the value of these coins is \$4.45, how many more nickels than quarters does John have?
5. If $a \star b = \dfrac{a+b}{a+3b}$, find the value of x that satisfies
$$2 \star (x \star 3) = 4$$
6. In how many ways can one arrange the letters of WINTER in such a way that the two vowels are never adjacent?
7. A theatre priced adult tickets to its play eight dollars higher than child tickets. At a certain performance, the theatre sold 75 more child tickets than adult tickets, for a total sales of \$1975. How much would the total sales have been if the prices of the child and adult tickets had been reversed?
8. How many zero's does the number $23! + 24!$ end in?
9. For how many integers k does the polynomial in x given by
$$4x^2 + kx - 9$$
factor over the integers?
10. Jorge has 57 coins with a total value of 90 cents. The coins are all pennies, nickels, or dimes. How many nickels does he have?
11. Six mathletes and two coaches sit at a circular table. If the two coaches sit across the table from each other, how many arrangements are possible?
12. Solve the system
$$\begin{cases} 13x + 14y &= 15 \\ 12x + 13y &= 14 \end{cases}$$
13. What is the ten's digit of 11^{2016}?

14. Determine the sum of all the solutions of the equation

$$|x-1|+|x|+|x+1|=|4x|$$

15. If r and s are the roots of $2x^2 = x + 10$, find the value of $r^2 + s^2$.

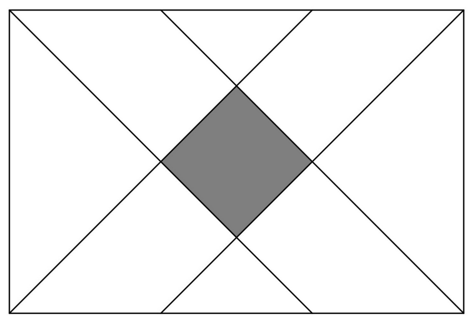
Fall 2016 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. In parallelogram $ABCD$, diagonals AC and BD intersect at point E. If $\angle BAC = 70°$ and $\angle BDC = 40°$, find $\angle AEB$.

2. In parallelogram $ABCD$, the angle bisectors of $\angle A$ and $\angle B$ meet at point E, which happens to lie on segment CD. Suppose that $AB = 20$ and that $\angle AEB$ is a right angle. Find AD.

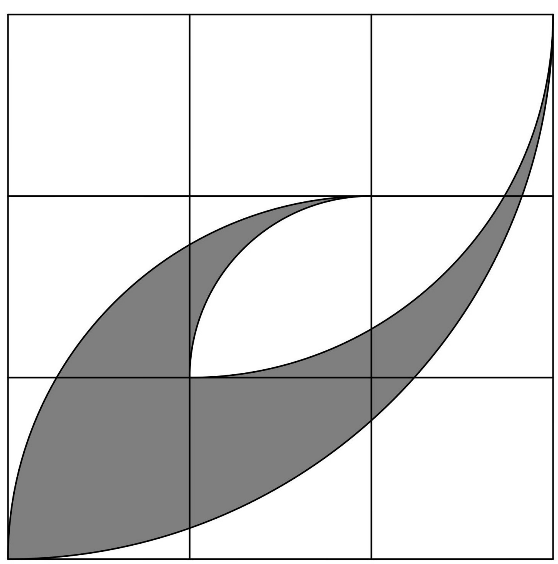
3. A rectangle has sides 4 and 6. The angle bisectors of each of the interior angles of the rectangle are drawn, forming a quadrilateral inside the rectangle. Find the area of this quadrilateral.



4. On a number line, the number $4/5$ ths of the way from a to $8/9$ is $7/9$. Find $3a$.

5. A convex polygon has exactly 135 diagonals. How many edges does it have?

6. In the figure, each arc is a quarter circle, for circles of radius 1, 2, and 3. What is the area of the shaded region enclosed by these arcs?



7. At each lattice point on the coordinate plane you can either step one unit up, one unit down, one unit left, or one unit right. In how many different ways can you start at the origin and in six such steps arrive at the point $(2, 2)$. You can revisit lattice points, including your final destination $(2, 2)$. Note: a point is a lattice point if both of its coordinates are integers.

8. Suppose that T is a regular tetrahedron of side length 1. An ant begins at the midpoint of an edge and wants to travel to the midpoint of the opposite edge. If the ant travels along the surface of the tetrahedron, what is the shortest possible distance in which the ant could make the trip?

9. Find the radius of the circumcircle of the triangle with side lengths 13, 14, and 15.

10. A trapezoid has its parallel bases of lengths 7 and 1. Find the length of the segment that is parallel to the bases and divides the trapezoid into two smaller trapezoids with equal areas.

11. Three congruent circles of radius 10 are mutually externally tangent. Find the area of the curved region in between the three circles.

12. An equilateral triangle ABC with side length 10 is inscribed in a circle. Diameter MN is drawn, parallel to BC and intersecting AB and AC at points P and Q respectively. Find QC.

13. A frustum has a larger radius of 5, a smaller radius of 2, and a height of 6. An inverted cone with base the same as the smaller base of the frustum and apex on the larger base of the frustum is removed from the frustum. Find the volume of the frustum that remains.

14. Eight points lie evenly spaced on a circle. Find the number of ways to draw four non-intersecting chords that join these points. Note that each of the eight points will be an endpoint of exactly one of these chords.

15. What is the maximum number of lattice points that can lie strictly inside an equilateral triangle of side length 7?

Fall 2016 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. How many two-digit integers are divisible by five?

2. Ruprecht is running North at a certain constant rate, while on the same road his friend Waldo is running South toward him at a constant rate that is 4 mph faster than one-third of Ruprecht's rate. If they started 40 miles apart and it took 2 hours and 15 minutes for them to meet, what is Waldo's rate?

3. Find the sum of all numbers which are two greater than their own reciprocal.

4. How many distinct numbers can be written in the form a/b, where $a \in \{1, 2, 3, 4, 5\}$ and $b \in \{5, 6, 7, 8, 9\}$?

5. Simplify to standard complex number form:
$$\frac{1}{i} + \frac{2}{i+1}$$

6. Find the coefficient of x^6 in the product $P(x) \cdot Q(x) \cdot R(x)$ where
$$\begin{aligned} P(x) &= (x+1)(x^4 + 2x^3 + 4x^2 + 8x + 16) \\ Q(x) &= (x-1)(x^2 - x - 1) \\ R(x) &= (x-2)(x^2 + x + 1) \end{aligned}$$

7. Let $\lfloor x \rfloor =$ the greatest integer $\leq x$. FInd all solutions to the equation
$$3x - 2\lfloor x \rfloor = 2$$

8. Find the sum of the reciprocals of the positive divisors of 1000.

9. In a magical forest, the cylindrical bamboo stalks increase their height at a rate of 4 inches per day. Their radius stays a constant value of $1/\sqrt{\pi}$ inch. One stalk of bamboo grows in every other (checkerboard-style) 3 inch by 3 inch square of a 10 foot by 10 foot plot of the forest. One day a panda comes across this plot of bamboo. On that day the bamboo started at a height of 4 feet. Due to the panda's considerable appetite, the rate of growth of the bamboo stalks' height is cut in half. In how many days will the volume of bamboo in this plot first reach a total volume of $50,000$ cubic inches? Count the day the panda arrives.

10. In how many ways can one arrange the letters of WINTER in such a way that the two vowels are never adjacent?

11. A particle starts at a fixed point P on a circle with center C. The particle moves on the circle, either 45° counterclockwise or 45° clockwise, each with probability 1/2. After eight such moves, the particle is at a point we call Q. What is the probability that $\angle PCQ$ is right? (Note: it is possible that points P and Q coincide.)

12. Calculate the remainder when 100^{1008} is divided by 2017.

13. Simplify

$$\sqrt[3]{99 - 70\sqrt{2}}$$

14. Suppose that the roots of the polynomial $x^3 + 2x^2 + 3x - 7$ are a, b, and c. Find a monic polynomial with coeffiecients in $\mathbb{Z}$ with roots ab, bc, and ac.

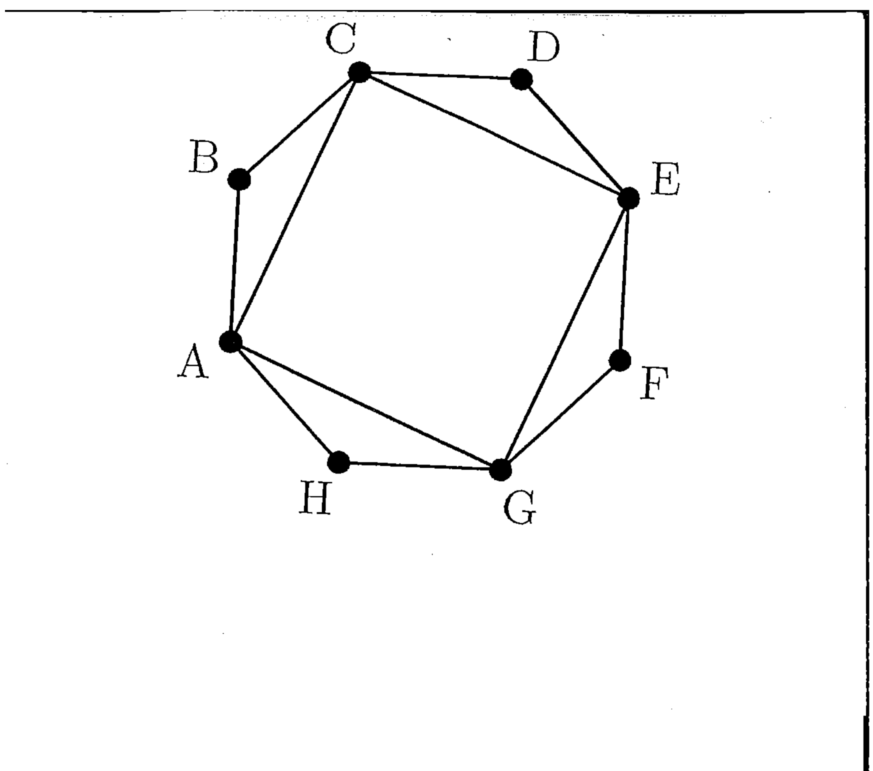
15. Factor completely over the integers:

$$x^8 + x^5 + x^3 + 1$$

Fall 2016 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. On a number line, the number $4/5$ ths of the way from a to $8/9$ is $7/9$. Find $3a$.
2. In parallelogram $ABCD$, diagonals AC and BD intersect at point E. If $\angle BAC = 70^\circ$ and $\angle BDC = 40^\circ$, find $\angle AEB$.
3. How many zero's occur when the number $2^{23} * 3 * 5^{24} * 7$ is written out in standard form?
4. Simplify to standard complex number form:
$$\frac{1}{i} + \frac{2}{i+1}$$
5. Suppose that $ABCDEFGH$ is a regular octagon of side length 1. What is the area of quadrilateral $ACEG$?



6. Find
$$\sin(1^\circ) \cdot \sin(2^\circ) \cdot \sin(3^\circ) \cdot \ldots \cdot \sin(179^\circ)$$
7. Solve the equation
$$\log_2(x) - \log_4(x+1) = 1$$
for x.

8. Find positive integers x, y such that $x^2 - 10y^2 = 1$

9. How many ordered triples of positive odd integers (m, n, p) satisfy

$$m + n + p = 21$$

?

10. Let $f(x) = \dfrac{3x+8}{2x+5}$ and $g(x) = \dfrac{3+2x}{3x}$. Find the value of $(f \circ g^{-1})^{-1}\left(\dfrac{11}{7}\right)$.

11. Find three odd positive integers a, b, and c, which satisfy

$$a^2 + b^2 + c^2 = 3abc$$
$$a + b + c = 19$$

12. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 17$.

13. Find all the real and complex roots of the polynomial

$$(x+1)^5 + (x+1)^4(x-1) + (x+1)^3(x-1)^2+ \\ + (x+1)^2(x-1)^3 + (x+1)(x-1)^4 + (x-1)^5$$

14. A sphere is inscribed in a cone. The ratio of the height of the cone to the radius of the cone is known to be $\sqrt{3}$. What fraction of the volume of the cone is taken up by the sphere?

15. Compute the sum

$$\sum_{k=0}^{\infty} \frac{k^2 - k}{2^{k+1}}$$

Fall 2016 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. Simplify to standard complex number form:
$$\frac{1}{i}+\frac{2}{i+1}$$

2. How many positive integers have a length of four digits in both base seven and base eight?

3. There exists a polynomial $Q(x)$ with integer coefficients such that
$$x^{12}+x^9+x^6+x^3+1=Q(x)\cdot(x^4+x^3+x^2+x+1)$$
Find the sum of the coefficients of $Q(x)$.

4. Bill has a magic bag containing red and blue stones. This bag has an interesting property—-whenever a stone is drawn, two stones of the other color appear in the bag! If Bill starts out with six red and four blue stones in the bag, what is the probability that his first four draws from the bag are all red? Note that the stones that Bill draws out are not put back in the bag.

5. Let $f(x)=\dfrac{3x+8}{2x+5}$ and $g(x)=\dfrac{3+2x}{3x}$. Find the value of $(f\circ g^{-1})^{-1}\left(\dfrac{11}{7}\right)$.

6. The value $\sin(75^\circ)$ can be expressed in the form
$$\frac{\sqrt{a}+\sqrt{b}}{c}$$
where a, b, and c are positive integers, and a and b have no perfect square factors greater than one. Find the product abc.

7. Randomly and independently chose two real numbers a and b from the interval $(0,1)$. What is the probability that $ab\le 3/8$?

8. How many real roots does $f'(x)$ have given that
$$f(x)=x(x+1)(x+2)(x+3)(x+4)$$
?

9. Let $n\in\mathbb{N}$. Find the n^{th} derivative of $f(n)=x^ne^x$ at $x=0$.

10. Consider the function
$$f(x)=\begin{cases}1+x & x\in\mathbb{Q}\\ e^x & x\notin\mathbb{Q}\end{cases}$$
What is its derivative at $x=0$?

11. A rocket is launched vertically at a constant speed of 250 meters per second. A camera is located 2000 meters horizontally from the launch pad. As soon as the rocket launches, the camera moves horizontally away from the launch pad at a rate of 50 meters per second. The camera is tracking the rocket and measuring the angle of elevation θ. Find $d\theta/dt$ when the rocket's altitude is 6000 meters.

12. Find the maximum possible value of y if $x^2 + xy + y^2 = 300$

13. Determine
$$\lim_{x \to \infty} \left(x^2 - x^4 \ln \left(\frac{1 + x^2}{x^2} \right) \right)$$

14. Let $f(x) = x^2 + 3x$ for $-1 \leq x \leq 1$. Let $g(x) = f^{-1}(x)$. Find the value of $g''(0)$.

15. At which x-coordinate does the graph of $y = e^x$ bend the most? For example, a parabola bends the most at its vertex. While an ellipse bends the most at the endpoints of its major axis.