Fall 2014 McNabb GDCTM Contest
Pre-Algebra

NO Calculators Allowed

1. Express 188 as the sum of two prime numbers.
2. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?
3. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks 4/3 as fast as Jill. When they meet, how far is Jack from where Jill started?
4. A number of students pitch in to buy a gift for their teacher. If each pays 8 dollars, the total collected would be too great by 3 dollars. If each pays 7 dollars, the total collected would be too little by 4 dollars. How much does the gift cost?
5. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.
6. Forty-one erasers are distributed to n students. If at least one student always receives at least 6 erasers no matter how the erasers are distributed, what is the largest possible value of n?
7. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?
8. Suppose that m and n are positive integers satisfying
$$\frac{1}{77} = \frac{n}{7} - \frac{3}{m}$$
Find the value of $m + n$.
9. Find the smallest possible value of $ab + cd + ef$ if each letter stands for a distinct element of the set $\{1, 2, 3, 4, 5, 6\}$.
10. There are 5 yellow balls, 8 red balls, and 7 green balls in a bag. What is the minimum number of balls that must be drawn to guarantee that at least 6 of them are the same color?
11. The pages of the book *Science of Mechanics in the Middle Ages* are numbered from 1 to 711. Considering all the digits needed to print these page numbers starting from page 1, on what page number does the 241st '1'occur?
12. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1 + \sqrt{3}, \qquad 2 - \sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

13. Find the sum of the positive even factors of 10000.
14. In how many ways can 4 different rings be placed on the four fingers of the right hand? Here the order of the rings on a given finger matters and each finger can accomodate all four rings.
15. Reading right to left, what is the first non-zero digit of 25!?

Fall 2014 McNabb GDCTM Contest
Algebra One

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?

2. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.

3. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?

4. Suppose that m and n are positive integers satisfying
$$\frac{1}{77} = \frac{n}{7} - \frac{3}{m}$$
Find the value of $m + n$.

5. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1 + \sqrt{3}, \qquad 2 - \sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

6. Recall that $\binom{m}{n}$ stands for the number of ways of choosing n objects from a set of m objects. Name a solution m greater than 2 of the equation
$$\binom{m+2}{3} = 4\binom{m}{2}$$

7. Simplify
$$x - y - (3x - 4y) - (x - y - (3x - 4y))$$

8. How many arrangements of the letters in GDCTM do not have any 3 consecutive letters in alphabetical order? So, for instance, you would count DGCTM but you would not count DGCMT.

9. Solve:
$$\Big| x - 3|x - 4| \Big| = |7 + 2x|$$

10. Suppose n is a positive integer greater than or equal to 10. For such n, define $f(n)$ to be sum of the units digit of n and twice the value of $f(m)$ where m is the integer that remains when the units digit of n is removed. In case n is a positive integer less than or equal to 9, define $f(n) = n$. Find the value of $f(1145)$.

11. Simplify $\left(\sqrt{6}^{\sqrt{12}}\right)^{\sqrt{3}}$.

12. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.

13. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 13$.

14. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?

15. The infinite expression

$$\sqrt{9 + \sqrt{9 + \sqrt{9 + \cdots}}}$$

can be written in the form $\dfrac{a + \sqrt{b}}{c}$ where a, b, and c are positive integers with no common factor greater than one. Find the value of $a + b + c$.

Fall 2014 McNabb GDCTM Contest
Geometry

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?

2. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks $4/3$ as fast as Jill. When they meet, how far is Jack from where Jill started?

3. A brigade of over a thousand men can line up in 13 rows of equal length with 4 soldiers left over and it can line up in 19 rows of equal length with 1 soldier left over. What is the smallest possible size of the brigade?

4. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1+\sqrt{3}, \qquad 2-\sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

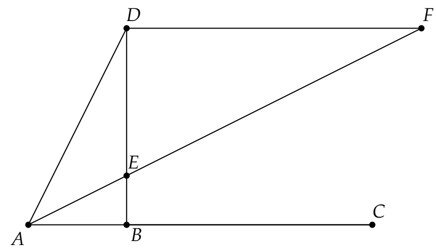
5. Solve:
$$\Big|x-3|x-4|\Big| = |7+2x|$$

6. A fountain has two basins, one above and one below, each of which has three outlets. The first outlet of the top basin fills the lower basin in one hour, the second in two hours, and the third in three hours. When all three upper outlets are shut, the first outlet of the lower basin empties it in two hours, the second in three hours, and the third in four hours. If all the outlets are opened, how long in hours will it take for the lower basin to fill?

7. For m a positive integer, let $g(m)$ be the number of distinct prime factors of m. For example, $g(12) = 2$. Find the value of $g(g(60) \cdot g(91))$.

8. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.

9. In the diagram, segment DB is perpendicular to both DF and AC, and $A - E - F$ are collinear. In addition, $EF = 2AD$. If $\angle FAC$ measures 17° find the angle measure in degrees of $\angle DAB$.



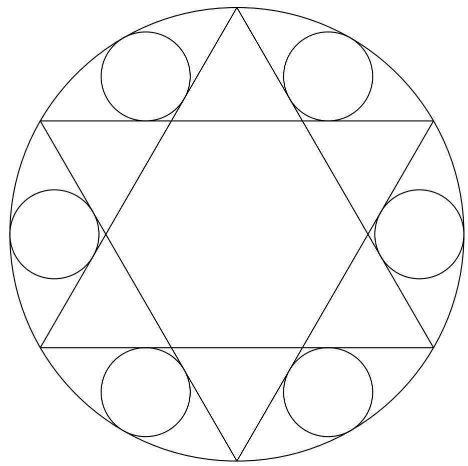
10. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?

11. The line $y = mx$ intersects the lines $x + y = 7$ and $x + y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?

12. In right $\triangle ABC$, its inscribed circle meets legs BA and BC at points D and E respectively. If $BD = 3$ and $DA = 11$, find the length of leg BC.

13. In $\triangle ABC$, point D lies on segment AB so that $AD/DB = 2/11$ while point E lies on segment BC so that $CE/EB = 3/7$. Let AE and CD intersect at point F. Find the ratio DF/FC.

14. A regular six-pointed star, composed of two intersecting equilateral triangles each of side length $6\sqrt{3}$, is inscribed in a circle. Six congruent smaller circles are internally tangent to this circle and externally tangent to the star. Find the radius of the small circles.



15. Find the volume of the tetrahedron with vertices located at $(0,0,0)$, $(1,-2,1)$, $(1,2,-1)$, and $(2,1,-1)$.

Fall 2014 McNabb GDCTM Contest
Algebra Two

NO Calculators Allowed

1. Fiji apples cost \$4.68 for a half-dozen and 90 cents a piece. Gala apples cost \$5.39 for a half-dozen and 97 cents a piece. If Sarah buys 8 Fiji apples and 9 Gala apples with a \$20 bill, how much change should she receive?

2. Forty-one erasers are distributed to n students. If at least one student always receives at least 6 erasers no matter how the erasers are distributed, what is the largest possible value of n?

3. Simplify
$$x - y - (3x - 4y) - (x - y - (3x - 4y))$$

4. The area of a rectangle is 864. The sum of the length and width is 60. By how much does the length exceed the width?

5. A regular hexagon is inscribed in a circle of radius one while another regular hexagon is circumcribed about this circle. What is the area of the region enclosed by the two hexagons?

6. How many arrangements of the letters in GDCTM do not have any 3 consecutive letters in alphabetical order? So, for instance, you would count DGCTM but you would not count DGCMT.

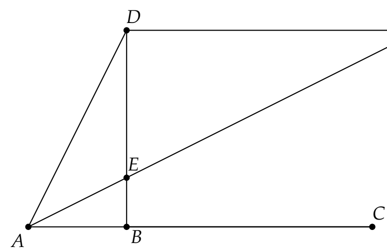
7. Solve the system
$$\begin{cases} \dfrac{x}{6} + \dfrac{4}{y} & = 2 \\ \dfrac{18}{x} + \dfrac{y}{2} & = 5 \end{cases}$$

8. Solve:
$$\Big| x - 3|x - 4| \Big| = |7 + 2x|$$

9. Simplify $\left(\sqrt{6}^{\sqrt{12}}\right)^{\sqrt{3}}$.

10. What is the first time after 1pm that the minute and hour hand of a clock will overlap? Answer to the nearest second and express your answer in hours:minutes:seconds format.

11. In the diagram, segment DB is perpendicular to both DF and AC, and $A - E - F$ are collinear. In addition, $EF = 2AD$. If $\angle FAC$ measures 17° find the angle measure in degrees of $\angle DAB$.



12. The line $y = mx$ intersects the lines $x + y = 7$ and $x + y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?

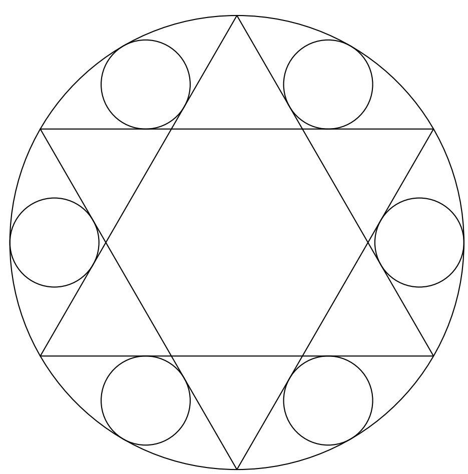
13. Find the value of

$$\sum_{k=1}^{100} i^{k(k+1)/2}$$

Here, i stands for the square root of negative one.

14. Find the sum of the real roots of the polynomial $x^4 - 16x^2 - 40x - 25$.

15. A regular six-pointed star, composed of two intersecting equilateral triangles each of side length $6\sqrt{3}$, is inscribed in a circle. Six congruent smaller circles are internally tangent to this circle and externally tangent to the star. Find the radius of the small circles.



Fall 2014 McNabb GDCTM Contest
PreCalculus

NO Calculators Allowed

1. Express 188 as the sum of two prime numbers.

2. Jack and Jill start walking toward each other. Initially they were 700 meters apart. Jack walks 4/3 as fast as Jill. When they meet, how far is Jack from where Jill started?

3. Lincoln Middle School has 324 students and 17 teachers. A field trip for the entire school (students and teachers) to the Kimball Museum is planned. Each of their buses holds at most 28 passengers. Each bus must have at least one teacher on it. What is the minimum number of buses the school requires for this field trip?

4. If a ounces of tea leaves brews b cups of tea and c cups fill one thermos, how many ounces of tea leaves must be brewed to fill d thermos's? Answer in terms of a, b, c, and d.

5. The pages of the book *Science of Mechanics in the Middle Ages* are numbered from 1 to 711. Considering all the digits needed to print these page numbers starting from page 1, on what page number does the 241st '1'occur?

6. Recall that $\binom{m}{n}$ stands for the number of ways of choosing n objects from a set of m objects. Name a solution m greater than 2 of the equation
$$\binom{m+2}{3} = 4\binom{m}{2}$$

7. Find the value of
$$\sum_{k=1}^{100} i^{k(k+1)/2}$$
Here, i stands for the square root of negative one.

8. Suppose n is a positive integer greater than or equal to 10. For such n, define $f(n)$ to be sum of the units digit of n and twice the value of $f(m)$ where m is the integer that remains when the units digit of n is removed. In case n is a positive integer less than or equal to 9, define $f(n) = n$. Find the value of $f(1145)$.

9. Let $\lfloor x \rfloor$ be the greatest integer which is less than or equal to x. Write the solution of the equation
$$\lfloor 2x \rfloor = \lfloor 3x \rfloor$$
using interval notation.

10. Define recursively the function $a(m,n)$:
$$\begin{aligned} a(m,n) &= a(m-1, a(m, n-1)) \\ a(0,n) &= n+1 \\ a(m,0) &= a(m-1, 1) \end{aligned}$$
Find the value of $a(2,2)$.

11. Find the volume of the tetrahedron with vertices located at $(0,0,0)$, $(1,-2,1)$, $(1,2,-1)$, and $(2,1,-1)$.

12. The line $y = mx$ intersects the lines $x + y = 7$ and $x + y = -14$ at points A and B respectively. If $AB = 39$, what is a possible value for the slope m?

13. In $\triangle ABC$ point D lies on side BC so that AD bisects angle BAC. If $AD = 13$, $DC = 37$, and $AC = 40$, find the length of AB.

14. Let $f(x)$ be a function which satisfies for all x and y the relation

$$f(x) \cdot f(y) - f(xy) = x + y$$

Determine the function $f(x)$.

15. In how many ways can 4 different rings be placed on the four fingers of the right hand? Here the order of the rings on a given finger matters and each finger can accomodate all four rings.

Fall 2014 McNabb GDCTM Contest
Calculus

NO Calculators Allowed

Assume all variables are real unless otherwise stated in the problem.

1. A number of students pitch in to buy a gift for their teacher. If each pays 8 dollars, the total collected would be too great by 3 dollars. If each pays 7 dollars, the total collected would be too little by 4 dollars. How much does the gift cost?

2. Write down in order from least to greatest (separate by commas) these irrational numbers:
$$1+\sqrt{3}, \qquad 2-\sqrt{2}, \qquad 2\sqrt{2}, \qquad \frac{\sqrt{2}}{2}$$

3. Solve the system
$$\begin{cases} \dfrac{x}{6}+\dfrac{4}{y} & = 2 \\ \dfrac{18}{x}+\dfrac{y}{2} & = 5 \end{cases}$$

4. In right $\triangle ABC$, its inscribed circle meets legs BA and BC at points D and E respectively. If $BD = 3$ and $DA = 11$, find the length of leg BC.

5. Find one positive value of b so that $x = \tan^{-1} b$ solves the equation
$$\tan^2(2x) + \tan^2(x) = 10$$

6. You are given three weighings involving twelve balls, of which eleven are the same weight but one is either heavier or lighter than the rest. The balls are numbered 1 through 12. The scale has two pans, a left and a right. When balls $1, 4, 7, 10$ are put in the left pan and balls $3, 6, 9, 12$ are put in the right pan, the left pan is heavier. When balls $3, 6, 9, 10$ are put in the left pan and balls $2, 5, 8, 12$ are put in the right pan, the left pan is lighter. When balls $3, 4, 8, 12$ are put in the left pan and balls $2, 6, 7, 11$ are put in the right pan, the right pan is heavier. Which ball is different and is it heavier or lighter than the rest?

7. Let the function y satisfy
$$\frac{dy}{dx} = y^2(y-2)(y+2)$$
If $y(0) = -1/5$ find the limit as $x \to \infty$ of $y(x)$.

8. Let
$$f(x) = \left(\sum_{k=0}^{100} x^k\right) \cdot \left(\sum_{k=0}^{100} (-1)^k x^k\right)$$
Determine the value of $f'(1)$.

9. Let f be differentiable for all x. Let $f(1) = 1/2$. Suppose for all $x \neq 0$ that $f(x) = f(1/x)$. Let $g(x) = x^4 f(x)$. Find the value of $g'(1)$.

10. Let a be a positive constant. If the minimum value of $f(x) = e^x + ae^{-x}$ is $3a$, find the value of a.

11. Determine
$$\lim_{x \to 0} (\cos x)^{\cot^2 x}$$

12. The infinite expression
$$\sqrt{9 + \sqrt{9 + \sqrt{9 + \cdots}}}$$
can be written in the form $\dfrac{a + \sqrt{b}}{c}$ where a, b, and c are positive integers with no common factor greater than one. Find the value of $a + b + c$.

13. Find the equation of a line which is tangent to $y = x^4 - 8x^2 + 3x + 5$ at two distinct points.

14. Let
$$f(x) = \begin{cases} e^{3x-(1/x^2)} + 2e^{3x} & \text{if } x \neq 0 \\ 2 & \text{if } x = 0 \end{cases}$$
Determine the value of $f'(0)$.

15. Find two positive rational numbers r and s, neither of which are integers, so that $r^2 + s^2 = 13$.