Spring 2014 GDCTM/McNabb Pre-Algebra Contest

NO Calculators Allowed

1. If $\angle A$ is supplementary to $\angle B$ and $\angle B$ is complementary to $\angle C$ then find the degree measure of $\angle A$ given that the sum of all three angles equals 217°.

2. Given that 3/4 of a number is 36/49 what is 1/4 of that number?

3. Find the sum of the first 20 positive odd integers.

4. Name a fraction less than 4/5 and greater than 3/4 whose denominator is less than twenty. Note that both the numerator and denominator of your fraction must be positive integers.

5. The sum of an integer, its cube, and its cube root is 522. What is this integer?

6. Two glasses of water together contain 29 ounces. One glass of water has 5 more ounces than the other. How much water does the larger glass have?

7. Find the value of
$$211 \cdot 253 + 147 \cdot 289 + 253 \cdot 289 + 147 \cdot 211$$

8. In how many way can the letters in $DALLAS$ be arranged? Include the original way!

9. Jessica has three standard six-sided dice—one is blue, one is red, and one is white. In how many ways can she roll the dice so that the sum of the dots showing on top equals 12?

10. Square the base-two number 111. Give your answer in base-two notation as well.

11. Let a, b, and c be distinct positive integers such that at least two of them are divisible by 6 and at least two of them are divisible by 9. Find the minimum possible value of $a + b + c$.

12. Find the number of factors of 30! which are perfect cubes.

13. Find the prime factorization of 9991.

14. Four cards are drawn randomly from a standard 52 card deck. What is the probability that no two of these cards belong to the same suit?

15. Eight cows graze a pristine field bare in 40 days. It would take 15 cows just 12 days to graze the same pristine field bare. How many days would it take 10 cows to graze that same pristine field bare? Assume that the grass in this field grows at a constant rate and the cows graze at a constant rate.

Spring 2014 GDCTM/McNabb Algebra One Contest

NO Calculators Allowed

1. In how many way can the letters in $DALLAS$ be arranged? Include the original way!
2. Jessica has three standard six-sided dice—one is blue, one is red, and one is white. In how many ways can she roll the dice so that the sum of the dots showing on top equals 12?
3. Let a, b, and c be distinct positive integers such that at least two of them are divisible by 6 and at least two of them are divisible by 9. Find the minimum possible value of $a+b+c$.
4. Find the number of factors of 30! which are perfect cubes.
5. How many positive integers less than 20 can be written as the sum of three consecutive integers?
6. For what value of the constant a does the equation $ax-3a+1=4(3x-a)-ax$ have no solution?
7. How many times do the graphs of the functions $f(x)=1000|x|$ and $g(x)=x^2/1000$ intersect?
8. Find all value(s) of the constant a so that the line $ax+2y=7$ is perpendicular to the line $ax-3y=2$.
9. Find the prime factorization of 9991.
10. Four cards are drawn randomly from a standard 52 card deck. What is the probability that no two of these cards belong to the same suit?
11. Hezy needs to buy 8 apples, 6 oranges, and 3 plums. He recalls that the prices per piece of these fruits are 27 cents, 72 cents, and 54 cents, but he does not remember which price goes with which fruit. What is the least amount in cents that Hezy must bring with him to the store to ensure that he can buy all the fruit he needs?
12. A running back tries to first elude tackler A and then tackler B. His probability of eluding A is $2/3$ and his probability of eluding both A and B is $1/2$. What is the probability that the running back eludes B once he has eluded A?
13. Find the sum of the square roots of the roots of the quadratic equation $x^2-39x+25=0$.
14. Solve the system

$$\begin{aligned} ab^2c^2 &= -18 \\ a^3bc^2 &= 12 \\ ab^2c &= -36 \end{aligned}$$

15. Eight cows graze a pristine field bare in 40 days. It would take 15 cows just 12 days to graze the same pristine field bare. How many days would it take 10 cows to graze that same pristine field bare? Assume that the grass in this field grows at a constant rate and the cows graze at a constant rate.

Spring 2014 GDCTM/McNabb Geometry Contest

NO Calculators Allowed

1. How many times do the graphs of the functions $f(x) = 1000|x|$ and $g(x) = x^2/1000$ intersect?

2. Find all value(s) of the constant a so that the line $ax + 2y = 7$ is perpendicular to the line $ax - 3y = 2$.

3. Four cards are drawn randomly from a standard 52 card deck. What is the probability that no two of these cards belong to the same suit?

4. A running back tries to first elude tackler A and then tackler B. His probability of eluding A is $2/3$ and his probability of eluding both A and B is $1/2$. What is the probability that the running back eludes B once he has eluded A?

5. Find the sum of the square roots of the roots of the quadratic equation $x^2 - 39x + 25 = 0$.

6. Solve the system

$$\begin{aligned} ab^2c^2 &= -18 \\ a^3bc^2 &= 12 \\ ab^2c &= -36 \end{aligned}$$

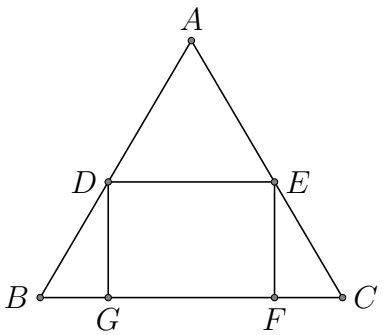
7. The area of a trapezoid is 24. If all of the sides of the trapezoid are doubled while keeping all the corresponding angles the same, what is the area of the new trapezoid?

8. Find the area of pentagon $ABCDE$ if the coordinates of the vertices are: $A = (0, 0), B = (1, -4), C = (9, 2), D = (0, 5), E = (-4, 2)$.

9. How many pairs of vertical angles are formed by 7 distinct lines all passing through the same point?

10. Eight cows graze a pristine field bare in 40 days. It would take 15 cows just 12 days to graze the same pristine field bare. How many days would it take 10 cows to graze that same pristine field bare? Assume that the grass in this field grows at a constant rate and the cows graze at a constant rate.

11. Suppose square $DEFG$ is inscribed in equilateral triangle ABC as shown. Find the ratio of the area of ABC to the area of $DEFG$.

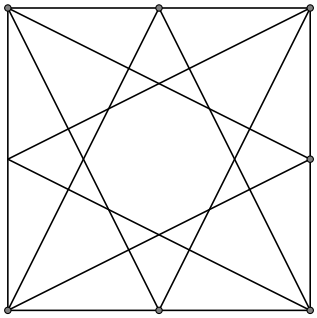


12. Nine colorless unit squares are assembled into a 3 by 3 square. Two squares are to be colored red and two others yellow. In how many different ways can this be done if two colorings are considered to be the same if one can be rotated to coincide with the other?

13. Point P can be any point on a fixed circle of radius seven, while point Q can be any point in the plane of the circle at a distance of three from point P. Find the area of the set of all possible locations of the point Q.

14. Recall that a *Pythagorean Triple* has the form (a, b, c) where a, b, and c are positive integers satisfying $a^2 + b^2 = c^2$. Find a Pythagorean Triple in which a or b equals 17.

15. In a square of side length six, segments are drawn from the midpoint of each side to the opposite vertices of the square, forming a convex octagon as shown below. Find the area of this octagon.



Spring 2014 GDCTM/McNabb Algebra Two Contest

NO Calculators Allowed

1. Point P can be any point on a fixed circle of radius seven, while point Q can be any point in the plane of the circle at a distance of three from point P. Find the area of the set of all possible locations of the point Q.

2. A baseball is hit so that its height in feet t seconds after impact is given by $h(t) = 80t - 16t^2 + 3$. What is the maximum height, in feet, this baseball reaches?

3. Find the smallest positive integer greater than 200 with exactly 8 positive factors.

4. If $\sec x - \tan x = 2$ find the value of $\sec x + \tan x$.

5. Four apples are weighed 2 at a time in all possible ways, giving these six weights in ounces: $16, 16, 18, 19, 21$, and 21. How many ounces do all four apples together weigh?

6. Factor $x^4 + 2x^3 - 15x^2 + 8x - 1$ into the product of two quadratic polynomials with integer coefecients.

7. An airplane makes a round trip flying the same route each way. Its speed (relative to the ground) is 480 mph in one direction while it is 600 mph in the other. What is the plane's average speed for the entire round trip?

8. How many ordered pairs of positive integers (x, y) satisfy the equation

$$xy + x + y = 2014$$

   ?

9. If $\dfrac{(a-b)(c-d)}{(b-c)(d-a)} = 4$ then find the value of $\dfrac{(a-c)(b-d)}{(a-b)(c-d)}$.

10. Eight cows graze a pristine field bare in 40 days. It would take 15 cows just 12 days to graze the same pristine field bare. How many days would it take 10 cows to graze that same pristine field bare? Assume that the grass in this field grows at a constant rate and the cows graze at a constant rate.

11. Find the sum of the square roots of the roots of the quadratic equation $x^2 - 39x + 25 = 0$.

12. In a right triangle, the sum of the legs is equal to 10 and the altitude from the right angle to the hypotenuse has length 3. Find the length of the hypotenuse.

13. If $x \ln x = 15$ then what is the least integer greater than x?

14. If $\log_a b = 2$ and $\log_b c = 4$, find the value of $\log_c a$.

15. Solve the equation

$$\sqrt[3]{x+8} + \sqrt[3]{8-x} = 1$$

Spring 2014 GDCTM/McNabb PreCalculus Contest

NO Calculators Allowed

1. A baseball is hit so that its height in feet t seconds after impact is given by $h(t) = 80t - 16t^2 + 3$. What is the maximum height, in feet, this baseball reaches?

2. If $\sec x - \tan x = 2$ find the value of $\sec x + \tan x$.

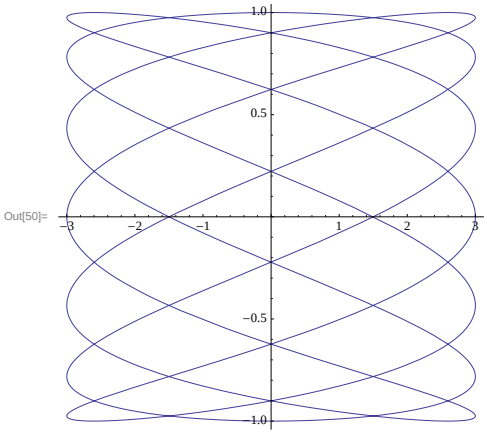
3. Factor $x^4 + 2x^3 - 15x^2 + 8x - 1$ into the product of two quadratic polynomials with integer coefecients.

4. Eight cows graze a pristine field bare in 40 days. It would take 15 cows just 12 days to graze the same pristine field bare. How many days would it take 10 cows to graze that same pristine field bare? Assume that the grass in this field grows at a constant rate and the cows graze at a constant rate.

5. If $x \ln x = 15$ then what is the least integer greater than x?

6. Solve the equation
$$\sqrt[3]{x+8} + \sqrt[3]{8-x} = 1$$

7. The graph of the parametric curve $x = a\cos bt$, $y = \sin 3t$ for $0 \le t < 2\pi$ is shown below. If a and b are positive integers find the value of ab.



8. If x solves
$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x+2014}$$
find the value of x^2.

9. Find the period of the function
$$f(x) = \frac{\cos(x/3) + \sec(x/2) + \cot(x/4)}{\sin(x/5) + \tan(x/8) + \csc(x/9)}$$

10. Let $a_1, a_2, a_3, \cdots$ be an infinite geometric sequence. Suppose that $\sum_{n=1}^{\infty} a_n = -9/20$ and $\sum_{n=4}^{\infty} a_n = 2/15$. Find the common ratio of this sequence.

11. At what point (x, y) in the coordinate plane does the curve $(t^3 - 16t, t^2 - 4)$ intersect itself?

12. Find the range of the function

$$f(x) = \frac{4}{x-4} - \frac{9}{x-9}$$

Give your answer in interval notation.

13. Quadrilateral $ABCD$ is inscribed in a circle with $AB = 5$, $BC = 7$, $CD = 6$ and $DA = 8$. Find the ratio AC/BD.

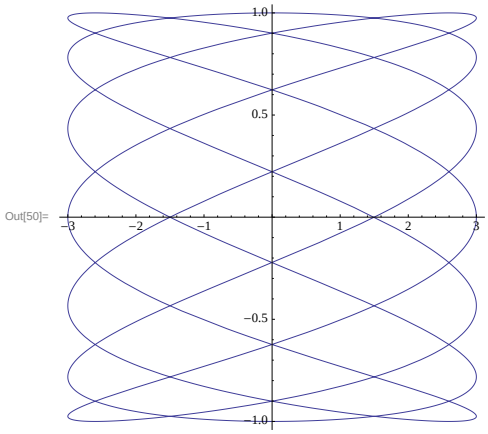
14. Let z and w be complex numbers satisfying $|z| = 1$ and $|w - 3 - 4i| = 1$. Find the maximum possible value of $|z + w|$.

15. A data center has 3 distinct servers, labeled A, B, and C. Each web request is routed independently to one of these servers. A request is routed to A with probability $1/2$, to B with probability $1/3$, and to C with probability $1/6$. What is the expected number of web requests that are received immediately before the first request that is routed to C?

Spring 2014 GDCTM/McNabb Calculus Contest

NO Calculators Allowed

1. The graph of the parametric curve $x = a\cos bt$, $y = \sin 3t$ for $0 \le t < 2\pi$ is shown below. If a and b are positive integers find the value of ab.



2. If x solves
$$\frac{1}{x} = \frac{1}{x+1} + \frac{1}{x+2014}$$
find the value of x^2.

3. Find the period of the function
$$f(x) = \frac{\cos(x/3) + \sec(x/2) + \cot(x/4)}{\sin(x/5) + \tan(x/8) + \csc(x/9)}$$

4. Let $a_1, a_2, a_3, \cdots$ be an infinite geometric sequence. Suppose that $\sum_{n=1}^{\infty} a_n = -9/20$ and $\sum_{n=4}^{\infty} a_n = 2/15$. Find the common ratio of this sequence.

5. Find the range of the function
$$f(x) = \frac{4}{x-4} - \frac{9}{x-9}$$
Give your answer in interval notation.

6. A data center has 3 distinct servers, labeled A, B, and C. Each web request is routed independently to one of these servers. A request is routed to A with probability $1/2$, to B with probability $1/3$, and to C with probability $1/6$. What is the expected number of web requests that are received immediately before the first request that is routed to C?

7. Let f be a continuous, strictly increasing function. If $\int_1^9 f(x)\,dx = 52$, $f(1) = 4$, and $f(9) = 8$, find the value of $\int_4^8 f^{-1}(x)\,dx$.

8. Find
$$\lim_{a\to 0} \frac{\int_0^a \ln(1+ax)\,dx}{a^3}$$

9. Let $f(x)$ be a function integrable on the interval $[-5, 5]$. If $\int_{-5}^3 f(x)\,dx = 8$ and $\int_0^3 f(x)\,dx = 1$, find the value of $\int_{-5}^0 3f(x)\,dx$.

10. Let g be a differentiable function on the interval $[-3, 3]$ so that $g(1) = 0$. If

$$f(x) = \int_0^{g(x)} \frac{1}{t^4 + 4}\, dt$$

find the value of $\lim_{x\to 1} \dfrac{g(x)}{f(x)}$.

11. Let $F(a) = \int_0^a x^3 - ax\, dx$ for $a \geq 0$. Find the minimum value of $F(a)$.

12. Evaluate the integral

$$\int_0^{\pi/3} \sin(5x)\sin(3x)\, dx$$

13. Define recursively the sequences x_n and y_n by

$$x_{n+1} = \frac{1}{3}x_n + \frac{2}{3}y_n$$
$$y_{n+1} = \frac{2}{3}x_n + \frac{1}{3}y_n$$

If $x_1 = 30$ and $y_1 = 40$ find $\lim_{n\to\infty}(x_n, y_n)$.

14. Let b be a real number and consider the cubic equation $x^3 + 3bx + 3 = 0$. The set of all real numbers b so that this cubic has three distinct real solutions is an interval of the form $(-\infty, r)$. The real number r has the form $r = -\sqrt[3]{\dfrac{a}{b}}$ where a and b are positive integers with no common factor greater than one. Find the value of $a + b$.

15. Evaluate the integral

$$\int_0^1 \frac{x^{\sqrt{2}} - 1}{\ln x}\, dx$$