Spring 2010 McNabb GDCTM Contest
Level IV

1. Amy is in a line of 40 people at the movies. Ahead of her stand 12 people. How many stand behind her?

   (A) 13 **(B)** 27 **(C)** 28 **(D)** 29 **(E)** 39

2. Let a, b, c, and d be positive real numbers such that $abc = 12$, $bcd = 6$, $acd = 125$, and $abd = 3$. Then $abcd$ equals

   (A) 12 **(B)** 15 **(C)** 30 **(D)** 45 **(E)** 60

3. Three pomegranates and one pineapple weigh as much as sixteen plums. Four plums and one pomegranate weight as much as one pineapple. How many pomegranates weigh as much as 3 pineapples?

   (A) 5 **(B)** 6 **(C)** 7 **(D)** 9 **(E)** 11

4. For $z = a + bi$ a complex number, it's conjugate is $\overline{z} = a - bi$. Let S denote the set of all complex numbers z so that the real part of $1/\overline{z}$ equals one. Then set S is

   (A) a line **(B)** a circle **(C)** a parabola
   (D) the empty set **(E)** an hyperbola

5. How many lines of symmetry does a cube have?

   (A) 4 **(B)** 7 **(C)** 10 **(D)** 12 **(E)** 13

6. If $\sin x + \cos x = a$, then $\sin 2x$ equals

   (A) $2a$ **(B)** $a^2 - 1$ **(C)** $1 - a^2$ **(D)** $a^2 + 1$ **(E)** $(a - 1)^2$

7. In throwing four fair cubical dice, what is the probability of obtaining two distinct doubles?

   (A) $\frac{5}{72}$ **(B)** $\frac{7}{36}$ **(C)** $\frac{1}{5}$ **(D)** $\frac{5}{16}$ **(E)** $\frac{3}{8}$

8. The function $f(x) = \dfrac{x + \sqrt{x^2 + 8}}{2}$ has domain $(-\infty, \infty)$ and on this domain is invertible. It's inverse function has the form $f^{-1}(x) = ax + \dfrac{b}{x}$ for some constants a and b. What is the value of $a + b$?

   (A) -1 **(B)** 0 **(C)** 1 **(D)** 2 **(E)** 3

9. A regular pentagon has each edge of length 2. Its area is closest to

(A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

10. Let k be a positive constant and let f be a continuous function on the interval $[-k, k]$. If $\int_{-k}^{k} f(x)\,dx = a$ then $\int_{-1}^{1} f(kx)\,dx$ equals

(A) a **(B)** ak **(C)** 1 **(D)** $\dfrac{k}{a}$ **(E)** $\dfrac{a}{k}$

11. Let f be continuously differentiable on the interval $[0, \pi]$. If $f(0) = 0$ and $f(\pi) = 0$, then

$$\int_0^{\pi} f(x)f'(x)\,dx$$

equals

(A) $-\pi$ **(B)** 0 **(C)** 1 **(D)** $\pi/2$ **(E)** cannot be uniquely determined

12. Suppose for every positive x that

$$xe^x = e + \int_1^{x^3} f(t)\,dt$$

Find the value of $f(8)$.

(A) $e/4$ **(B)** e^2 **(C)** $e^2/4$ **(D)** $3e^2$ **(E)** 6

13. Find the area inside the ellipse given by

$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

(A) 5π **(B)** 6π **(C)** $25\pi/4$ **(D)** 7π **(E)** 8π

14. Let $\sum_{n=1}^{\infty} a_n$ be a positive term convergent series. Which of the following series must converge?

$$\text{I.} \quad \sum_{n=1}^{\infty} \frac{1}{a_n}$$

$$\text{II.} \quad \sum_{n=1}^{\infty} \sqrt{a_n}$$

$$\text{III.} \quad \sum_{n=1}^{\infty} a_n^2$$

(A) I only **(B)** II only **(C)** III only **(D)** II and III only **(E)** I, II, and III

15. Let $I_n = \int_0^1 x^n e^{-x}\, dx$, where n is a positive integer. Which of the following is true?

$$\begin{array}{ll} \text{I.} & I_n = -e^{-1} + nI_{n-1} \quad \text{for } n \geq 2 \\ \text{II.} & I_n = n! - [en!]e^{-1} \quad \text{for } n \geq 2 \\ \text{III.} & \lim_{n\to\infty} I_n = 0 \end{array}$$

where the notation $[x]$ stands for the greatest integer less than or equal to x.

(A) I only **(B)** II only **(C)** III only **(D)** I and III only **(E)** I, II, and III

16. What is the coefficient of x^{10} in the expansion of

$$(1+x)(1+x^2)(1+x^3)\cdots(1+x^{10})$$

(A) 9 **(B)** 10 **(C)** 11 **(D)** 12 **(E)** 32

17. A 4 inch by 4 inch square board is subdivided into sixteen 1 inch by 1inch squares in the usual way. Four of the smaller squares are to be painted white, four black, four red, and four blue. In how many different ways can this be done if each row and each column of smaller squares must have one square of each color in it? (The board is nailed down: it can not be rotated or flipped).

(A) 576 **(B)** 864 **(C)** 1152 **(D)** 1200 **(E)** 1600

18. A cubic polynomial $P(x)$ satisfies $P(1) = 1$, $P(2) = 3$, $P(3) = 5$, and $P(4) = 6$. Then the value $P(7)$ must equal

(A) 10 **(B)** 7 **(C)** 0 **(D)** -3 **(E)** -7

19. Hezy tosses a fair cubical die until each number 1 through 6 has come up at least once. On average, Hezy requires X tosses to do this. The real number X is closest to

(A) 12.8 **(B)** 14.7 **(C)** 16.3 **(D)** 17.2 **(E)** 19.5

20. In $\triangle ABC$, points D, E, and F are located on $\overline{BC}$, $\overline{AC}$, and $\overline{BA}$ respectively, so that $\overline{AD}$, $\overline{BE}$, and $\overline{CF}$ are concurrent at point P, the area of $\triangle BPD$ is 190, the area of $\triangle DPC$ is 380, and the area of $\triangle CPE$ is 418. Then the area of $\triangle APE$ is

(A) 121 **(B)** 143 **(C)** 242 **(D)** 319 **(E)** 330